Section 1

August 2020 − 2 months

1 (a) Redemption proceeds $= 1.2 \times 20\,000 \times \dfrac{\text{RPI (June 2020)}}{\text{RPI (Base date)}}$

RPI (June 2020) is used because there is a time lag of 2 months.

August 2020 − 2 months = June 2020

$$\Rightarrow 1.2 \times 20\,000 \times \frac{213.4(1.05^{11})}{192.2}$$

$$= £45\,575.83$$

(b)

Date	Time lag	Payment
15/8/2009	15/6/2009	$1000 \times \dfrac{213.4}{192.2}$
15/8/2010	15/6/2010	$1000 \times \dfrac{213.4(1.05)}{192.2}$
⋮	⋮	⋮
15/8/2020	15/6/2020	$(20\,000 \times 1.2 + 1000) \times \dfrac{213.4(1.05^{11})}{192.2}$

$$P = 1000\left(\frac{213.4}{192.2}\right)v^{\frac{1}{2}} + 1000\left(\frac{213.4(1.05)}{192.2}\right)v^{3/2} + \dots$$

$$\dots + 25000\left(\frac{213.4(1.05^{11})}{192.2}\right)v^{11.5}$$

$$= \frac{1000(213.4)}{192.2}v^{\frac{1}{2}}\left[1 + \frac{1.05}{1.10} + \frac{1.05^2}{1.10^2} + \dots + \frac{1.05^{10}}{1.10^{10}}\right]$$

24000
+1000

$$+ \frac{25000(213.4 \times 1.05^{11})}{192.2(1.10)^{11.5}}$$

Define a new interest rate $j = \frac{1.10}{1.05} - 1 = 0.04762$

$$\Rightarrow \frac{1000(213.4)}{192.2(1.10)^{\frac{1}{2}}}\ddot{a}_{\overline{11}|}^{\,j} + \frac{25000(213.4 \times 1.05^{11})}{192.2(1.10^{11.5})}$$

$$= 4328.45 + 15865.27$$

$$= £25193.72$$

2 (a) $1+i_n \sim \log N(\mu, \sigma^2)$

$E(i_n) = 0.08 \qquad Var(i_n) = 0.04$

Referring to the yellow tables, we obtain:

$1.08 = e^{\mu + \frac{1}{2}\sigma^2}$

$0.04 + 1.08^2 = e^{2\mu + 2\sigma^2}$

$\mu + \frac{1}{2}\sigma^2 = \ln(1.08)$ ——— (1)

$\mu + \sigma^2 = \frac{1}{2}\ln(0.04 + 1.08^2)$ ——— (2)

Solve these two simultaneous equations to obtain values for μ and σ

(2) − (1)

$\frac{1}{2}\sigma^2 = 0.016859 \qquad \sigma^2 = 0.033719$

$\sigma = 0.18363 \qquad \mu = 0.060102$

(b) $S_{10} = \prod_{k=1}^{10} (1+i_k)$

$S_{10} = (1+i_1)(1+i_2) \ldots (1+i_{10})$

$\ln S_{10} = \ln\left(\prod_{k=1}^{10}(1+i_k)\right) = \ln(1+i_1) + \ln(1+i_2) + \ldots + \ln(1+i_{10})$

$\ln S_{10} \sim N(0.060102 \times 10, \; 0.033719 \times 10)$

This gives

$\ln S_{10} \sim N(0.60102, \; 0.33719)$

(c) $P(S_{10} < 0.75)$

Take logarithms in both sides to give:

$P(\ln S_{10} < \ln 0.75)$

$$= P\left(\frac{\ln S_{10} - 0.60102}{\sqrt{0.33719}} < \frac{\ln(0.75) - 0.60102}{\sqrt{0.33719}}\right)$$

$\approx$ by the Central Limit Theorem $P(Z < -1.530)$

$= 1 - 0.93699 = 0.0630$)

Section 2

1 (a) $AV(10) = 100\exp\left\{ \int_0^{10} \delta(u)\,du \right\}$ where $\delta(u)$ is the force of interest

$$= 100\exp\left\{ \int_0^5 \delta(u)\,du + \int_5^{10} \delta(u)\,du \right\}$$

$$= 100\exp\left\{ 0.02(5) + 0.07(5) \right\}$$

$$= £156.83$$

(b) $\rho(t) = 500$

Using the formula $PV(0) = \int_8^{10} \rho(t)\exp\left\{ -\int_0^t \delta(u)\,du \right\} dt$

$$PV(0) = \int_8^{10} 500\, \underbrace{e^{-\int_0^5 \delta(u)\,du - \int_5^t \delta(u)\,du}}_{\substack{\text{the discount factor is split into} \\ \text{two parts based on the force of interest defined} \\ \text{in the question}}} dt$$

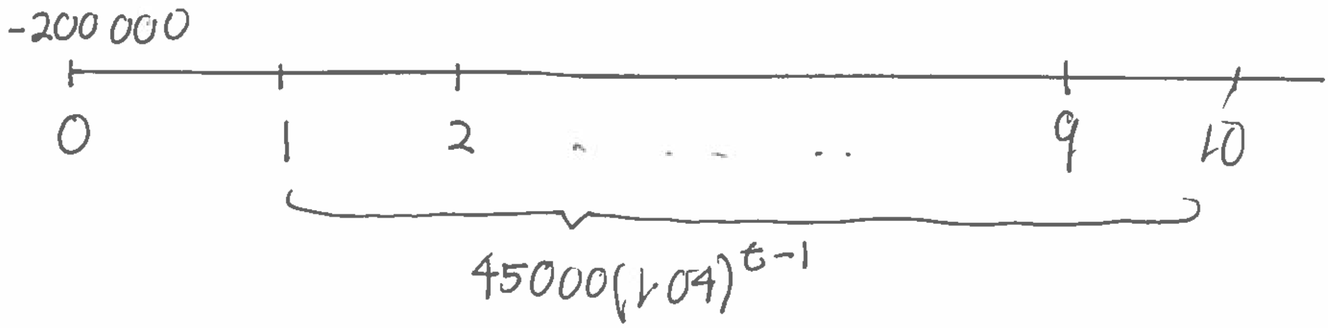
$$= \int_8^{10} 500 e^{-0.02(5) - 0.07(t-5)}\, dt$$

$$= 500 e^{0.25} \int_8^{10} e^{-0.07t}\, dt$$

$$= 500 e^{0.25} \left[\frac{e^{-0.07t}}{-0.07} \right]_8^{10}$$

$$= £684.42$$

2(a)



$$NPV(0) = -200\,000 + \int_1^{10} 45\,000(1.04)^{t-1}\,1.06^{-t}\,dt$$

$$= -200\,000 + \int_1^{10} \frac{45\,000}{1.04}(1.04)^{t}(1.06)^{-t}\,dt$$

Note: $1.04^{t} = e^{t\ln(1.04)}$

$1.06^{-t} = e^{-t\ln(1.06)}$

$1.04^{t}\cdot 1.06^{-t} = e^{-t(\ln(1.06)-\ln(1.04))}$

$$\Rightarrow -200\,000 + \int_1^{10} \frac{45000}{1.04} e^{-t(\ln(1.06)-\ln(1.04))}\,dt$$

$$= -200\,000 + \frac{45000}{1.04}\left[-\frac{e^{-t(\ln(1.06)-\ln(1.04))}}{\ln(1.06)-\ln(1.04)}\right]_1^{10}$$

$$= -200\,000 + 351\,119.12$$

$$= £151\,119.12$$

(b) PV of income $= \int_1^S 45000(1.04)^{t-1}\,1.06^{-t}\,dt$

$$= \frac{45000}{1.04}\int_1^S 1.04^t . 1.06^{-t}\,dt$$

$$= \frac{45000}{1.04}\int_1^S e^{-t(\ln(1.06)-\ln(1.04))}\,dt$$

$$= \frac{45000}{1.04}\left[-\frac{e^{-t(\ln(1.06)-\ln(1.04))}}{\ln(1.06)-\ln(1.04)}\right]_1^S$$

$$= \frac{45000}{1.04}\left[-\frac{e^{-S(0.0190482)}}{0.0190482}+51.50788\right]$$

Equate the PV of income to the PV of cost and solve for S

$$\frac{45000}{1.04}\left[-\frac{e^{-S(0.0190482)}}{0.0190482}+51.50788\right]=200\,000$$

$$e^{-0.0190482S} = 0.893087$$

$$S = 5.94$$

Section 3

1) (a) $$\bar{\tau} = \frac{\int_0^{20} t\,v(0,t)\,dt}{\int_0^{20} v(0,t)\,dt}$$ [definition of duration]

$$= \frac{\int_0^{20} t(1.05)^{-t}\,dt}{\int_0^{20}(1.05)^{-t}\,dt} = \frac{\int_0^{20} te^{-t\ln(1.05)}\,dt}{\int_0^{20} e^{-t\ln(1.05)}\,dt}$$

Looking at the numerator,

$$\int_0^{20} te^{-t\ln(1.05)}\,dt = \left[\frac{-te^{-t\ln(1.05)}}{\ln(1.05)}\right]_0^{20} + \int_0^{20}\frac{e^{-t\ln(1.05)}}{\ln(1.05)}\,dt$$

integration by parts

$$= \frac{-20e^{-20\ln 1.05}}{\ln(1.05)} - \left[\frac{e^{-t\ln(1.05)}}{(\ln(1.05))^2}\right]_0^{20}$$

$$= 107.264$$

$$\int_0^{20} e^{-t\ln(1.05)}\,dt = \left[-\frac{1}{\ln(1.05)}e^{-t\ln(1.05)}\right]_0^{20}$$

$$= 12.77123$$

$$\bar{\tau} = \frac{107.264}{12.77123} = 8.40 \text{ years}$$

(b) To be immunised, the duration of assets must equal the duration of the liabilities. The government bond has a duration of 4 years which is approximately half the duration of the insurance company's liabilities.

(c) Start with the LHS

$$\int_0^n t^2 e^{-\delta t}\,dt = -\frac{1}{\delta} e^{-\delta t} \cdot t^2 \Big|_0^n + \frac{1}{\delta}\int_0^n 2t e^{-\delta t}\,dt$$

integration by parts

$$= \frac{-n^2 e^{-\delta n}}{\delta} + \frac{2}{\delta}\int_0^n t e^{-\delta t}\,dt$$

The definition for $(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t e^{-\delta t}\,dt$

Using this gives:

$$\Rightarrow \frac{-n^2 v^n}{\delta} + \frac{2}{\delta}(\bar{I}\bar{a})_{\overline{n}|}$$

$$= \frac{2(\bar{I}\bar{a})_{\overline{n}|} - n^2 v^n}{\delta}$$

(d) You need to check each of the following conditions

1. $NPV_A = NPV_L$

2. $\tau_A = \tau_L$

3. $conv_A > conv_L$

Starting with ①

$$NPV_A = 12.61(1.05^{-5}) + 7.68(1.05^{-20})$$
$$= 12.77$$

$$NPV_L = 12.77$$

First condition is satisfied

②

$$\bar{\tau}_A = \frac{12.61(5)(1.05^{-5}) + 7.68(20)(1.05^{-20})}{12.77}$$
$$= 8.40 \text{ years}$$

From part (a), we know $\bar{\tau}_L = 8.40$ years.

2nd condition is satisfied.

③

$$conv_A = \frac{12.61(25)(1.05^{-5}) + 7.68(20^2)(1.05^{-20})}{12.77}$$
$$= 110.01$$

this comes from $t^2(1.05^{-t})$

$$conv_L = \frac{\int_0^{20} t^2 e^{-t\ln(1.05)}\,dt}{12.77}$$

Looking at the numerator

$$\int_0^{20} t^2 e^{-t\ln(1.05)} \stackrel{\text{part (c)}}{=} \frac{2\left(\frac{\bar{a}_{\overline{20|}} - 20(1.05^{-20})}{\delta}\right) - 20^2(1.05^{-20})}{\delta}$$

where $\delta = \ln(1.05)$ and $\bar{a}_{20} = \dfrac{1-1.05^{-20}}{\ln(1.05)}$

Substituting this in gives

$\Rightarrow 1307.08$

$$\text{conv}_L = \frac{1307.08}{12.77} = 102.36$$

The third condition is satisfied.

2 (a) Denote the security price at time t by S_t

Then $S_0 = S$

To answer these types of questions, you will need to consider two portfolios:

Portfolio A: Long position in the forward contract.

Portfolio B: Hold 1 unit of stock

Invest $-Ke^{-\delta T}$ at the risk free rate.

At time T, the payoffs are

Portfolio A : $-K + S_T$

Portfolio B : $-Ke^{-\delta T} e^{\delta T} + S_T = -K + S_T$

Since the payoffs are equal at time T, by the no arbitrage principle, the portfolio values at $t=0$ must also be identical.

At $t=0$:

Portfolio A has value 0

Portfolio B has value $-Ke^{-\delta T} + S_0$

Equating these two gives: $K = S_0 e^{\delta T}$

$K = Se^{\delta T}$

b) At $t=0$: $K_0 = S_0 e^{\delta_0 \times 1.5} = 9.50 e^{1.5 \times 0.06} = £10.39$

At $t=1$: $K_1 = S_1 e^{\delta_1 \times 0.5} = 7.60 e^{0.055(0.5)} = £7.81$

Value at $t=1$ is $(K_0 - K_1) e^{-\delta_1 \times 0.5}$

$$= (10.39 - 7.81) e^{-0.055 \times 0.5}$$

$$= £2.51$$

The value is £2.51 to the short position counterparty.

Section 4

1(a) To solve this question, you will require

$$(1+f_{1,5})^5 = \frac{(1+y_6)^6}{(1+y_1)} \quad \text{—— (1)}$$

$$(1+f_{1,2})^2 = \frac{(1+y_3)^3}{(1+y_1)} \quad \text{—— (2)}$$

From equation (2)

$$1.05^2(1+y_1) = (1+y_3)^3 \quad \text{—— (3)}$$

$$(1+f_{3,3})^3 = \frac{(1+y_6)^6}{(1+y_3)^3}$$

Substituting $f_{3,3} = 0.07$ gives

$$1.07^3(1+y_3)^3 = (1+y_6)^6 \quad \text{—— (4)}$$

Substituting equations (4) and (3) into (1) gives

$$(1+f_{1,5})^5 = \frac{1.07^3(1+y_3)^3}{(1+y_3)^3} \times 1.05^2$$

$$= 0.061954687$$

(b) Using $P_6 = (1+y_6)^{-6}$

$$0.65 = \frac{1}{(1+y_6)^6}$$

$$y_6 = 0.074437$$

$$(1+f_{3,3})^3(1+y_3)^3=(1+y_6)^6$$

$$(1+y_3)^3=\frac{1.074437^6}{1.07^3}$$

$$(1+f_{1,2})^2(1+y_1)=(1+y_3)^3$$

$$y_1=\frac{\frac{1.074437^6}{1.07^3}}{1.05^2}-1$$

$$=0.1390865$$

2(a) Expectations theory is when bond yields are determined by the investors expectations of future short-term interest rates - Thus bond returns are equal to expected returns from a series of short term investments

(b) Liquidity preference

Investors prefer short-term investments to long term investments. Thus, long term investments provide a yield to maturity greater than that implied by expectations of future short-term interest rates

Market segmentation

Segments of the yield curve reflect different supply and demand from the market.