SAS Portfolio Theory and Asset Models Solutions

Section 1

1 (a) For First order Stochastic Dominance, you require

$$F_A(x) \leq F_B(x) \quad \text{for all } x$$

or

$$F_B(x) \leq F_A(x) \quad \text{for all } x$$

In this case:

$$F_A(150) = 0.25 > F_B(150) = 0$$

However, there is a contradiction to the above statement

$$F_A(450) = 0.75 < F_B(450) = 1$$

Therefore, there is no first order Stochastic Dominance

(b) For Second Order Stochastic Dominance ~~you~~ you require:

$$\int_{-\infty}^{x} F_A(y)\,dy \leq \int_{-\infty}^{x} F_B(y)\,dy$$

or

$$\int_{-\infty}^{x} F_B(y)\,dy \leq \int_{-\infty}^{x} F_A(y)\,dy$$

To solve this question, you will first need the distribution functions of A and B

$$F_A(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{600} & 0 \leq x \leq 600 \\ 1 & x > 600 \end{cases}$$

$$F_B(x) = \begin{cases} 0 & \text{for } x < 200 \\ \frac{1}{2} & 200 \le x \le 400 \\ 1 & x > 400 \end{cases}$$

To show Second order Stochastic Dominance, you will need to compute integrals for the following intervals:

$x < 0$, $0 \le x \le 200$, $200 \le x \le 400$, $400 \le x \le 600$, $x > 600$.

Note: If there is a contradiction, you do not need to continue computing integrals.

$$\int_{-\infty}^{x} F_A(y)\,dy = 0 = \int_{-\infty}^{x} F_B(y)\,dy \quad [\text{For } x < 0]$$

$$\int_{-\infty}^{x} F_A(y)\,dy = \frac{x^2}{1200} > \int_{-\infty}^{x} F_B(y)\,dy = 0 \quad [0 \le x \le 200]$$

For $200 \le x \le 400$

$$\int_{-\infty}^{x} F_A(y)\,dy = \frac{x^2}{1200} > \int_{-\infty}^{x} F_B(y)\,dy = 0.5(x-200)$$

For $400 \le x \le 600$

$$\int_{-\infty}^{x} F_A(y)\,dy = \frac{x^2}{1200} > \int_{-\infty}^{x} F_B(y)\,dy = 100 + (x-400)$$

For $x > 600$

$$\int_{-\infty}^{x} F_A(y)\,dy = x - 300 = \int_{-\infty}^{x} F_B(y)\,dy$$

Since $\int_{-\infty}^{x} F_A(y)\,dy \geq \int_{-\infty}^{x} F_B(y)\,dy$ for all x, we can say that B is

2^{nd} order Stochastically dominant over A

c) Let the certainty equivalent be K.

$$u(K+w) = E((u(B)))$$
$$= 0.5\,u(200) + 0.5\,u(400)$$
$$= 0.5(1-e^{-0.01(200)}) + 0.5(1-e^{-0.01(400)})$$
$$= 0.923172$$

Now $u(K+w) = 1-e^{-0.01K}$

$$1-e^{-0.01K} = 0.923172$$
$$K = £256.62$$

Therefore, the certainty equivalent for B is £256.62

(d) Absolute risk aversion function:

$$A(x) = -\frac{u''(x)}{u'(x)}$$
$$u(x) = 1-e^{-0.01x}$$
$$u'(x) = 0.01e^{-0.01x}$$
$$u''(x) = -(0.01)^2 e^{-0.01x}$$
$$A(x) = -\frac{-(0.01)^2 e^{-0.01x}}{0.01e^{-0.01x}} = 0.01$$

e) $R(x) = \frac{-xu''(x)}{u'(x)}$

$= 0.01x$

f) The absolute risk aversion function is more appropriate as the risk of the random payment is not proportional to the wealth of the agent.

The relative risk aversion function is used when considering an investment problem; payout random variable is proportional to the initial wealth of the agent.

g) $u(k+w) = E(u(B+w))$ where k is the certainty equivalent and w is the initial wealth

$= 0.5u(400) + 0.5u(600)$

$= 0.5(1-e^{-4}) + 0.5(1-e^{-6})$

$= 0.9896028$

$\Rightarrow 1 - e^{-0.01(200+k)} = 0.9896028$

$200 + k = 456.62$

$k = 256.62$

The certainty equivalent is £256.62

Section 2

(a) You will need to find the loss distribution

$$E(Loss) = \frac{1}{3}\left(-8\% + (-10\%) + (-3\%)\right) = -7\%$$

$$Var(Loss) = \left(\frac{1}{3}\right)^2\Big(\underset{\substack{\uparrow \\ Var(A)}}{36} + \underset{\substack{\uparrow \\ Var(B)}}{64} + \underbrace{2(0.5)(6)(8)}_{2\rho\sigma_A\sigma_B}\Big) = \frac{148}{9}\%\%$$

$$Loss \sim N\left(300\,000 \times (-7\%),\ 300\,000^2 \times \left(\frac{148}{9}\%\%\right)\right)$$

$$95\%\ VaR = 300\,000(-7\%) + 300\,000(1.65)\sqrt{\frac{148}{9}}$$

$$= -987.71$$

Since the VaR is less than 0, we can say:

$$95\%\ VaR = 0$$

(b) $$E(R_p) = 1.08\pi_A + 1.1(1-\pi_A)$$
$$= 1.08\pi_A + 1.1 - 1.1\pi_A$$
$$= 1.1 - 0.02\pi_A$$

(c) $$\sigma_p^2 = \sigma_A^2\pi_A^2 + \sigma_B^2(1-\pi_A)^2 + 2\pi_A(1-\pi_A)\rho\sigma_A\sigma_B$$
$$= 36\pi_A^2 + 64(1-\pi_A)^2 + 2\pi_A(1-\pi_A)(0.5)(6)(8)$$
$$= 36\pi_A^2 + 64 - 128\pi_A + 64\pi_A^2 + 48\pi_A - 48\pi_A^2$$
$$= 52\pi_A^2 - 80\pi_A + 64$$

(6)

(a) To find the minimum variance portfolio, differentiate the above equation and set to 0

$$\sigma_P^2 = 52\pi_A^2 - 80\pi_A + 64$$

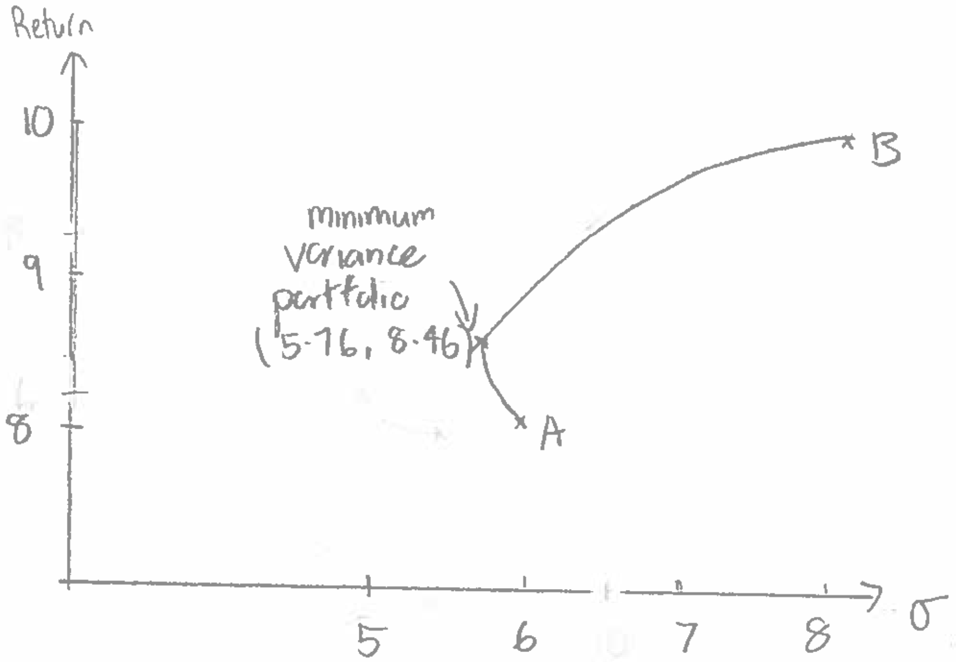
$$\frac{\partial \sigma_P^2}{\partial \pi_A} = 104\pi_A - 80$$

$$\Rightarrow 104\pi_A = 80$$

$$\pi_A = \frac{80}{104} = 0.7692$$

$$\pi_B = 1 - \pi_A = 0.2308$$

Hence : $(\pi_A, \pi_B) = (0.7692, 0.2308)$



Note: Opportunity set formed by 2 assets is just a curve

Variance of ~~minimum~~ minimum variance portfolio = 33.23

sd(min variance portfolio) = 5.76

Return = $1.1 - 0.02(0.7692) = 1.0846$

$= 8.46\%$

Section 3

(a) SML is:

$$E(R_i) - r_f = \beta_i (E(R_m) - r_f)$$

(b) We put the parameters for asset A and B into the SML equation and solve

Asset A: $8\% - r_f = 0.4(E(R_m) - r_f)$

Asset B: $12\% - r_f = 0.8(E(R_m) - r_f)$

$$8\% = 0.4E(R_m) + 0.6r_f \quad \text{—— } (1)$$

$$12\% = 0.8E(R_m) + 0.2r_f \quad \text{—— } (2)$$

$(1) \times 2$: $16\% = 0.8E(R_m) + 1.2r_f$ —— (3)

(3) − (2)

$$r_f = 4\%$$

$$E(R_m) = \frac{8\% - 0.6(4\%)}{0.4} = 14\%$$

c) If Asset A lies on the CML, it's return has the same expected return and standard deviation as a portfolio ~~be~~ formed by the risk free asset and the market portfolio.

CML equation: $E(R_A) - r_f = \left(\frac{E(R_m) - r_f}{\sigma_m}\right)\sigma_A$

If asset A lies on the CML:

$$8\% - 4\% = \left(\frac{14\% - 4\%}{\sigma_m}\right)(10\%) \Rightarrow \sigma_m = 25\%$$

d) The formula required is

$$\beta_i = \frac{\rho_{im}\sigma_i}{\sigma_m}$$

$$\beta_A = \frac{\rho_{AM}\sigma_A}{\sigma_m}$$

$$0.4 = \frac{\rho_{AM}(10\%)}{25\%}$$

$$\rho_{AM} = 1$$

e) Use the same formula as above ~~for~~ but use asset ~~A~~ B instead of asset A

$$\beta_B = \frac{\rho_{BM}\sigma_B}{\sigma_m}$$

$$0.8 = \frac{\rho_{BM}(25\%)}{25\%}$$

$$\rho_{BM} = 0.8$$

Section 4

(a) Since $R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \varepsilon_i$, taking expectations gives:

$$E(R_i) = a_i + b_{i1} E(I_1) + b_{i2} E(I_2)$$

Therefore:

$$E(R_A) = 3 + (1)(2) + (0.5)(3) = 6.5\%$$

$$E(R_B) = 4 + (2)(2) + (1)(3) = 11\%$$

$$Var(R_A) = b_{A1}^2 \, Var(I_1) + b_{A2}^2 \, Var(I_2) + 2 b_{A1} b_{A2} \sigma_{12} + \sigma^2_{\varepsilon_A}$$

$$= (1)^2(169) + 0.5^2(196) + (2)(1)(0.5)(91) + 120$$

$$= 424\%\%$$

$$Var(R_B) = (2)^2(169) + (1)^2(196) + (2)(2)(1)(91) + 150$$

$$= 1386\%\%$$

$$Cov(R_A, R_B) = Cov(3 + I_1 + 0.5 I_2 \, , \, 4 + 2I_1 + I_2)$$

$$= Cov(I_1, 2I_1) + Cov(0.5 I_2, I_2) + Cov(I_1, I_2) + Cov(0.5 I_2, 2I_1)$$

$$= 2(\overset{169}{\cancel{91}}) + 0.5(196) + 91 + 0.5(91)(2)$$

$$= 618\%\%$$

b) For I_1' and I_2' to be orthogonal, we require $cov(I_1', I_2') = 0$ (10)

$$cov(I_1', I_2') = cov\left(I_1, I_2 - \frac{\sigma_{12}}{\sigma_1^2} I_1\right)$$

$$= cov(I_1, I_2) - \frac{\sigma_{12}}{\sigma_1^2} cov(I_1, I_1)$$

$$= 0$$

(c) $R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \varepsilon_i$

Now substitute in expressions for these

$$R_i = a_i + b_{i1} I_1' + b_{i2}\left(I_2' + \frac{\sigma_{12}}{\sigma_1^2} I_1'\right) + \varepsilon_i$$

$$= a_i + b_{i1} I_1' + b_{i2} I_2' + b_{i2} \frac{\sigma_{12}}{\sigma_1^2} I_1' + \varepsilon_i$$

$$= a_i + \left(b_{i1} + b_{i2} \frac{\sigma_{12}}{\sigma_1^2}\right) I_1' + b_{i2} I_2' + \varepsilon_i$$

Therefore

$$R_i' = a_i' + b_{i1}' I_1' + b_{i2}' I_2' + \varepsilon_i'$$

with

$a_i' = a_i$

$b_{i1}' = b_{i1} + b_{i2} \frac{\sigma_{12}}{\sigma_1^2}$

$b_{i2}' = b_{i2}$

$\varepsilon_i' = \varepsilon_i$

For Asset A

$a'_A = 3$

$b'_{A1} = 1 + 0.5\left(\frac{91}{169}\right) = 1.2692$

$b'_{A2} = 0.5$

For Asset B

$a'_B = 4$

$b'_{B1} = 2 + (1)\left(\frac{91}{169}\right) = 2.5385$

$b'_{B2} = 1$