Students' Actuarial Society

Stochastic Processes – Solutions

Section 1

1. $\{S_n\}_{n\geq 0}$ is a simple random walk with probability of an upward movement given by $p = 3/5$ and $S_0 = 4$.

   (a) We can consider the probability of being above a certain threshold based on the required number of upward movements. If we wish to be above 5 after 9 steps and are currently at 4 then we need a net gain of at least one upward movement, i.e. at least 5 up and no more than 4 down in those 9 steps:

$$\begin{aligned}\Pr(S_9 \geq 5) &= \Pr(Take\ at\ least\ 5\ upward\ steps\ in\ first\ 9\ steps)\\ &= \sum_{k=5}^{9} \binom{9}{k} \cdot \left(\frac{3}{5}\right)^k \cdot \left(\frac{2}{5}\right)^{9-k}\\ &= 0.7334\ (4\ d.p.)\end{aligned}$$

Equivalently you could consider the probability of going no more than 4 steps down which is available directly in the Cambridge Statistical Tables.

For the normal approximation we first consider the expected value and variance of S_{90}:

$$E[S_{90}] = S_0 + 90 * E[Jump] = 4 + 90 * \left[\left(\frac{3}{5}\right)(1) + \left(\frac{2}{5}\right)(-1)\right] = 22$$

$$Var[S_{90}] = 90 * Var[Jump] = 90 * \left[\left(\frac{3}{5}\right)(1)^2 + \left(\frac{2}{5}\right)(-1)^2 - \left(\frac{1}{5}\right)^2\right] = 86.4$$

Therefore we have $S_{90} \dot{\sim} N(22, 86.4)$. We then use this to approximate the probability:

$$\begin{aligned}\Pr(S_{90} \geq 45) &= \Pr\left(\frac{S_{90} - 22}{\sqrt{86.4}} \geq \frac{45 - 22}{\sqrt{86.4}}\right)\\ &\approx \Pr(Z \geq 2.4744)\\ &= 1 - \Pr(Z \leq 2.4744)\\ &\approx 1 - 0.993\\ &= 0.007\end{aligned}$$

   (b) To find this probability we consider $S_0 = 14$ instead of $S_0 = 4$ and now calculate the probability of ruin, i.e. $\Pr(there\ exists\ n > 0\ such\ that\ S_n \leq 0)$:

$$\begin{aligned}\Pr(exists\ n > 0\ such\ that\ S_n \leq -10) &= \Pr(exists\ n > 0\ such\ that\ S_n \leq 0 | S_0 = 14)\\ &= \Pr(Ruin | S_0 = 14)\end{aligned}$$

$$= \min\left(1, \frac{q^{14}}{p}\right)$$
$$= \left(\frac{q}{p}\right)^{14}$$
$$= \left(\frac{2/5}{3/5}\right)^{14}$$
$$= 0.00343\ (5\ d.p.)$$

2. Initial capital, $S_0 = 15$.
   Yearly income $= 5$.

   We pay out claims in year i according to the random variable C_i. This leaves us with total capital in year n:

$$S_n = 15 + \sum_{i=1}^{n} X_i$$

   Where the X_i represent the total gain in wealth for the insurance company in each year and follows the distribution:

$$X_i = \begin{cases} -10 & w.p.\,1/17 \\ -5 & w.p.\,2/17 \\ 0 & w.p.\,10/17 \\ 5 & w.p.\,4/17 \end{cases}$$

   a. In order to find the required bounded probability we first note that this example makes use of the scaling property of random walks, with a scalar of 5. We can therefore define a new variable $\tilde{S}_n$:

$$\tilde{S}_0 = 3, \qquad \tilde{X}_i = \begin{cases} -2 & w.p.\,1/17 \\ -1 & w.p.\,2/17 \\ 0 & w.p.\,10/17 \\ 1 & w.p.\,4/17 \end{cases}, \qquad \tilde{S}_n = \tilde{S}_0 + \sum_{i=1}^{n} \tilde{X}_i$$

   We then use the coupling property of random walks to define a new variable $\tilde{S}'_n$ by grouping together the two lowest values of $\tilde{X}_i$:

$$\tilde{S}'_0 = 3, \qquad \tilde{X}'_i = \begin{cases} -1 & w.p.\,3/17 \\ 0 & w.p.\,10/17, \\ 1 & w.p.\,4/17 \end{cases} \qquad \tilde{S}'_n = \tilde{S}'_0 + \sum_{i=1}^{n} \tilde{X}'_i$$

   Since $\tilde{S}'_n \geq \tilde{S}_n$ for all n, it is less likely for the modified process $\tilde{S}'_n$ to fall into negative values in comparison with $\tilde{S}_n$, hence:

$$\Pr\left(Ruin\ in\ \tilde{S}'_n\right) \leq \Pr(Ruin\ in\ \tilde{S}_n)$$

   We can find the probability of ruin for $\tilde{S}'_n$:

$$\Pr\left(Ruin\ in\ \tilde{S}'_n\right) = \left(\frac{q}{p}\right)^a = \left(\frac{3/17}{4/17}\right)^3 = \left(\frac{3}{4}\right)^3$$

   Therefore we have the required bound, $\Pr(Ruin\ in\ S_n) \geq \left(\frac{3}{4}\right)^3$.

b. i. We estimate $\hat{\pi} = \frac{number\ times\ company\ runs\ out\ of\ capital}{total\ simulations} = \frac{17}{100} = 0.17$

ii. The above estimate is an example of binomial counting, $X \sim Bin(100, 0.17)$, where X is the number of times the company runs out of capital.

$Var[X] = 100(0.17)(1 - 0.17) = 14.11$

By the original equation for π we have $\hat{\pi} = X/100$.

Using the above formula we have:

$$Var[\hat{\pi}] = Var\left(\frac{X}{100}\right) = \frac{1}{100^2} Var(X) = \frac{1}{100^2} \times 14.11 = 1.411 \times 10^{-3}$$

Therefore, $sd[\hat{\pi}] = \sqrt{1.411 \times 10^{-3}} = 0.03756$

iii. We want $S.D.[X] \leq 0.1$, let a be the number of simulations:

$$\frac{\sqrt{a(0.17)(0.83)}}{a} \leq 0.1 \rightarrow \ a^{-0.5} * 0.3756 \leq 0.1 \rightarrow \ a^{-1} \leq 0.07088$$
$$\rightarrow \ a \geq 14.11$$

Therefore we require at least 15 simulations for the required standard deviation.

Section 2

1) This question is a typical Poisson processes question which makes use of the superposition theorem.

a. We consider the total number of passengers to arrive as a Poisson process of rate $(2 + 1)$ per minute by the superposition theorem.

Let $N(t) =$ the number of customers who arrive in a period of t minutes. Thus $N(t) \sim Pois\ (3t)$ distribution.

$\Pr(in\ a\ 2\ minute\ period, the\ combined\ total\ number\ of\ passengers\ is\ at\ least\ 2)$ can be re-expressed in terms of $N(2)$ as $\Pr(N(2) \geq 2)$:

$$\begin{aligned}
\Pr(N(2) \geq 2) &= 1 - \Pr(N(2) < 2) \\
&= 1 - (\Pr(N(2) = 0) + \Pr(N(2) = 1) \\
&= 1 - \left(\frac{6^0 e^{-6}}{0!} + \frac{6^1 e^{-6}}{1!}\right) \\
&= 0.9826\ (4dp)
\end{aligned}$$

b. Let $X_n =$ the number of females in n total passengers. Thus $X_n \sim Bin(n, 2/3)$

$$\begin{aligned}
&\Pr(3\ females\ and\ 2\ males \mid 5\ arrive\ in\ total) \\
&= \Pr(X_5 = 3) \\
&= \binom{5}{3} (2/3)^3 (1/3)^2 \\
&= \frac{80}{243}
\end{aligned}$$

c. We know that the total number of passengers arrive with a Poisson process of rate 3 per minute. The superposition theorem then tells us that each passenger in the process is female with probability 2/3 independently of all else.

Let $X_n =$ the number of females in n total passengers. Thus $X_n \sim Bin(n, 2/3)$

$$\begin{aligned}
&\Pr(Third\ female\ arrives\ before\ second\ male)\\
&= \Pr(Within\ 4\ passengers, at\ least\ 3\ are\ female)\\
&= \Pr(X_4 \geq 3)\\
&= \Pr(X_4 = 3) + \Pr(X_4 = 4)\\
&= \binom{4}{3}(2/3)^3(1/3) + \binom{4}{4}(2/3)^4\\
&= \frac{16}{27}
\end{aligned}$$

2)

a. By the splitting theorem, cars that can pick up the hitchhiker will follow a PP with rate $200 * 0.02 = 4$ per hour, and cars which will not pick up the hitchhiker will follow a PP with rate $200 * 0.98 = 196$ per hour.

Let $T =$ waiting time until the hitchhiker is picked up. We then want $E[T]$.

The waiting time between occurrences in a PP follows an exponential distribution with parameter equal to the rate of the PP. Therefore $T \sim Exp(4)$.

$$E[T] = \frac{1}{4} hours = 15\ minutes$$

b. <u>Method1</u>

Let $N(t)$ be the number of cars which pass the hitchhiker without stopping to pick them up in a period of t hours $\sim Pois(196t)$. We then want to calculate $E[N(T)]$ where T is as above. Since T is in itself a random variable we need to use the conditional expectation approach:

$$E[N(T)] = \int_0^\infty E[N(T)|T = t] * f_T(t)dt = \int_0^\infty E[N(t)] * 4e^{-4t}dt$$

Where the above second step comes from using the probability density function for an exponential distribution with parameter 4 as in the case of T. The change in expectation comes from the splitting theorem giving two independent processes, with $N(t)$ coming from the process relating to cars not stopping, and T coming from the process relating to cars stopping – hence the components exhibit independence. The 2 Poisson Process needs to be mutually independent, otherwise we can't use this method.

$$\begin{aligned}
E[N(T)] &= \int_0^\infty 196t * 4e^{-4t}dt\\
&= 784\int_0^\infty te^{-4t}dt\\
&= 784\left\{\left[-\frac{1}{4}te^{-4t}\right]_0^\infty + \frac{1}{4}\int_0^\infty e^{-4t}dt\right\}\\
&= 784\left\{0 + \frac{1}{4}\left[-\frac{1}{4}e^{-4t}\right]_0^\infty\right\}\\
&= 784\left(\frac{1}{16}\right)\\
&= 49
\end{aligned}$$

Instinctively we should expect this answer as we expect to wait 15 minutes until a car stops and we expect 196 cars per hour not to stop – i.e. $196/4 = 49$ cars not to stop in 15 minutes.

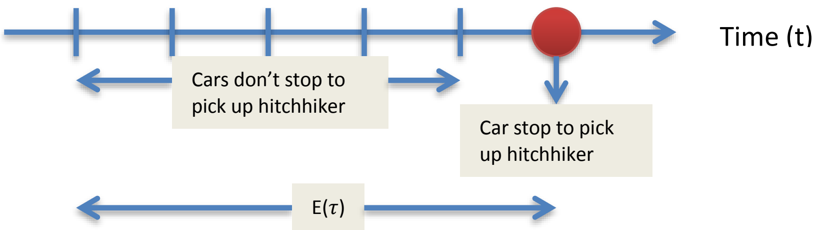
Method 2

We let T = waiting time until the hitchhiker is picked up where $T \sim Exp(4)$.
Let N(t) = the total number of cars that pass by the hitchhiker in a period of t hours.

By **superposition theorem**, we know that $\boldsymbol{N(t) = N_1(t) + N_2(t)}$ where $N_1(t) \sim Pois(\lambda_1)$ and $N_2(t) \sim Pois\ (\lambda_2)$. We define $N_1(t)$ as the number of cars which pass the hitchhiker without stopping to pick them up in a period of t hours and $N_2(t)$ as the number of cars which pass the hitchhiker and stopped to pick them up in a period of t hours.

Let τ=no. of cars that pass before a car stops and picks up the hitchhiker where $\tau \sim Geo(p_2)$. From part (a), we know that p_2= 0.02.

To calculate the expected number of cars that will pass the hitchhiker before they are picked up, refer to the diagram below:



So, the expected number that will pass the hitchhiker before they are picked up = E(τ) – 1

$$= \frac{1}{p_2} - 1$$

$$= \frac{1}{0.02} - 1$$

= 49 cars

Section 3

1) In this question we use the following labels, 1 = The Grassmarket, 2 = The Three Sisters, 3 = Rush, 4 = The Hive.

The transition matrix is given below, with columns and rows i corresponding to the i^{th} label as defined above:

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

2) This question is essentially asking about absorption probabilities with regards the states 1 and 6.

a. The transition matrix is given by:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The system of equations refers to the absorption probabilities derived from the above transition matrix – and concerns overall absorption into state 6.

Note $\{1\}$ is a closed class, $\{2,3,4,5\}$ is an open class, $\{6\}$ is a closed class.

$\alpha_1^{(6)} = 0$ since $\{1\}$ is a closed class and can never reach $\{6\}$ to be absorbed within it.
$\alpha_6^{(6)} = 1$ since we are already absorbed within the required class.

$$(1)\ \alpha_2^{(6)} = \frac{1}{2}\alpha_1^{(6)} + \frac{1}{2}\alpha_3^{(6)} = \frac{1}{2}\alpha_3^{(6)}$$
$$(2)\ \alpha_3^{(6)} = \frac{1}{2}\alpha_2^{(6)} + \frac{1}{2}\alpha_4^{(6)}$$
$$(3)\ \alpha_4^{(6)} = \frac{1}{3}\alpha_3^{(6)} + \frac{1}{3}\alpha_5^{(6)} + \frac{1}{3}\alpha_6^{(6)} = \frac{1}{3}\alpha_3^{(6)} + \frac{1}{3}\alpha_5^{(6)} + \frac{1}{3}$$
$$(4)\ \alpha_5^{(6)} = \alpha_4^{(6)}$$

- Why {2,3,4,5} as a class is needed?
  State 2, 3, 4 and 5 intercommunicate with each other but they do not communicate with other states in the Markov Chain thus they form a open class on it's own.

b. The desired probability is given by $\alpha_4^{(6)}$. We find it by solving the above system of equations. Note that $\sum_i \alpha_i^{(6)}$ does not need to sum to one, it is in fact the probabilities $\alpha_i^{(1)} + \alpha_i^{(6)} = 1\ \forall i$.

Subbing (4) into (3) we have:

$$\alpha_4^{(6)} = \frac{1}{3}\alpha_3^{(6)} + \frac{1}{3}\alpha_4^{(6)} + \frac{1}{3} \ \rightarrow\ \alpha_4^{(6)} = \frac{1}{2}\alpha_3^{(6)} + \frac{1}{2}$$

Subbing (1) into (2) we have:

$$\alpha_3^{(6)} = \frac{1}{2}\left(\frac{1}{2}\alpha_3^{(6)}\right) + \frac{1}{2}\alpha_4^{(6)} \ \rightarrow\ \alpha_3^{(6)} = \frac{2}{3}\alpha_4^{(6)}$$

Now sub this new formula for $\alpha_3^{(6)}$ into the above equation:

$$\alpha_4^{(6)} = \frac{1}{2}\left(\frac{2}{3}\alpha_4^{(6)}\right) + \frac{1}{2} = \frac{1}{3}\alpha_4^{(6)} + \frac{1}{2} \ \rightarrow\ \alpha_4^{(6)} = 3/4$$

3) This question is a simple probability question and can be answered without much (if any) stochastic processes knowledge. However it is helpful in the understanding of the principles behind branching processes.

Let $T = \#$ steps required to reach state 5. We therefore want $E[T]$. At most you can take four steps to reach state five and at best you can reach state five in one step.

Therefore T can take values $\{1,2,3,4\}$ and each of these values have a corresponding probability.

$$\begin{aligned}
\Pr(T=1) &= \Pr(Jump\ from\ state\ one\ to\ state\ 5) \\
&= \frac{1}{4} \\
\Pr(T=2) &= \Pr(1 \to 2 \to 5\ OR\ 1 \to 3 \to 5\ 0R\ 1 \to 4 \to 5) \\
&= \frac{1}{4} * \frac{1}{3} + \frac{1}{4} * \frac{1}{2} + \frac{1}{4} * 1 \\
\Pr(T=3) &= \Pr(1 \to 2 \to 3 \to 5\ OR\ 1 \to 3 \to 4 \to 5\ OR\ 1 \to 2 \to 4 \to 5) \\
&= \frac{1}{4} * \frac{1}{3} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2} * 1 + \frac{1}{4} * \frac{1}{3} * 1 \\
\Pr(T=4) &= \Pr(1 \to 2 \to 3 \to 4 \to 5) \\
&= \frac{1}{4} * \frac{1}{3} * \frac{1}{2} * 1
\end{aligned}$$

This then gives the expected value:

$$E[T] = \sum_i t_i \Pr(T = t_i) = \frac{25}{12}$$

Section 4

a. Because $q_{HD} = q_{SD} = 1$, there is a constant rate to die which does not depend on the state you are in (either healthy or sick). Therefore the time to death is exponentially distributed at that rate. This gives:

$$P_{HD}(t) = P_{SD}(t) = 1 - e^{-t}$$

The forward differential equation gives:

$$\begin{aligned}
P'_{HH}(t) &= P_{HH}(t) * q_{HH} + P_{HS}(t) * q_{SH} + P_{HD}(t) * q_{DH} \\
&= -6 * P_{HH}(t) + 6P_{HS}(t)
\end{aligned}$$

We also have that $P_{HH}(t) + P_{HS}(t) + P_{HD}(t) = 1$ which therefore gives us:

$$P_{HS}(t) = 1 - P_{HH}(t) - 1 + e^{-t}$$

We then substitute this back into the forward differential equation:

$$P'_{HH}(t) = -6P_{HH}(t) + 6(1 - P_{HH}(t) - 1 + e^{-t}) = -12P_{HH}(t) + 6e^{-t}$$

$$\to\ P'_{HH}(t) + 12P_{HH}(t) = 6e^{-t}$$

Now multiply both sides by e^{12t} as this will aid in the differential (we aim to turn the left hand side into a derivative function and then integrate both sides – thus removing the differentiated term):

$$P'_{HH}(t)e^{12t} + 12P_{HH}(t)e^{12t} = 6e^{11t}$$

We then notice that this can be expressed as a derivative as required:

$$[P_{HH}(t)e^{12t}]' = 6e^{11t}$$

Integrating both sides gives:

$$P_{HH}(t)e^{12t} = \int 6e^{11}tdt = \frac{6}{11}e^{11t} + C$$

We know the boundary condition $P_{HH}(0) = 1$ and we use this to find the constant, C:

$$P_{HH}(0) = 1 = \frac{\frac{6}{11}e^{11(0)} + C}{e^{12(0)}} \rightarrow C = 1 - \frac{6}{11} = \frac{5}{11}$$

Therefore:

$$P_{HH}(t) = \left(\frac{6}{11}e^{11t} + \frac{5}{11}\right)e^{-12t} = \left(\frac{6}{11} + \frac{5}{11}e^{-11t}\right)e^{-t}$$

If we then substitute the above into the previously developed equation linking $P_{HH}(t)$ to $P_{HS}(t)$ we obtain:

$$\begin{aligned} P_{HS}(t) &= 1 - P_{HH}(t) - 1 + e^{-t} \\ &= e^{-t} - P_{HH}(t) \\ &= e^{-t} - \frac{6}{11}e^{-t} - \frac{5}{11}e^{-12t} \\ &= \frac{5}{11}e^{-t} - \frac{5}{11}e^{-12t} \\ &= \left(\frac{5}{11} - \frac{5}{11}e^{-11t}\right)e^{-t} \end{aligned}$$

In addition we have already found that:

$$P_{HD}(t) = 1 - e^{-t}$$

We have therefore proved the three required probability statements.

Also, to conclude the first result, when $t \rightarrow \infty$, $e^{-t} \rightarrow 0$ then $p_{HD} \rightarrow 1$. From this, we know that from healthy state to dead state has probability of 1 when t approaches ∞.

b. Calculating a jumping chain matrix is simple. It follows two simple rules:
   - If $q_{ii} < 0$ then $p_{ij}^* = -\frac{q_{ij}}{q_{ii}}$ for $i \neq j$ and $p_{ii}^* = 0$.
   - If $q_{ii} = 0$ then $p_{ij}^* = 0$ for $i \neq j$ and $p_{ii}^* = 1$.

   This gives:

$$P^* = \begin{pmatrix} 0 & 5/6 & 1/6 \\ 6/7 & 0 & 1/7 \\ 0 & 0 & 1 \end{pmatrix}$$