

Section 1

To solve questions in this section, you will need to know the properties of <sup>the</sup> distribution function and density function.

Properties of the cdf

(a)  $0 \leq F_X(x) \leq 1$  for all  $x \in \mathbb{R}$

(b)  $\lim_{x \rightarrow \infty} F_X(x) = 1$  ,  $\lim_{x \rightarrow -\infty} F_X(x) = 0$

(c)  $F_X(x)$  is a non decreasing function.

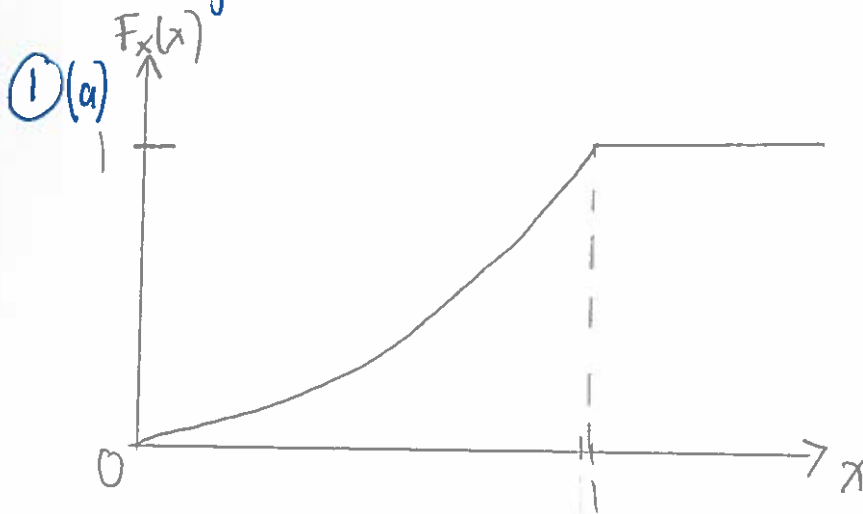
(d)  $F_X(x)$  is right continuous with left limits

Properties of the density function

(a) The area under the graph for a density function must equal to 1

(b)  $f_X(x) \geq 0$  for  $x \in \mathbb{R}$

(c) Image of the function is defined at all places where  $f_X(x) > 0$



From the graph above, it is obvious that all the 4 properties for the cdf are satisfied so  $F_X(x)$  is a distribution function

(2)

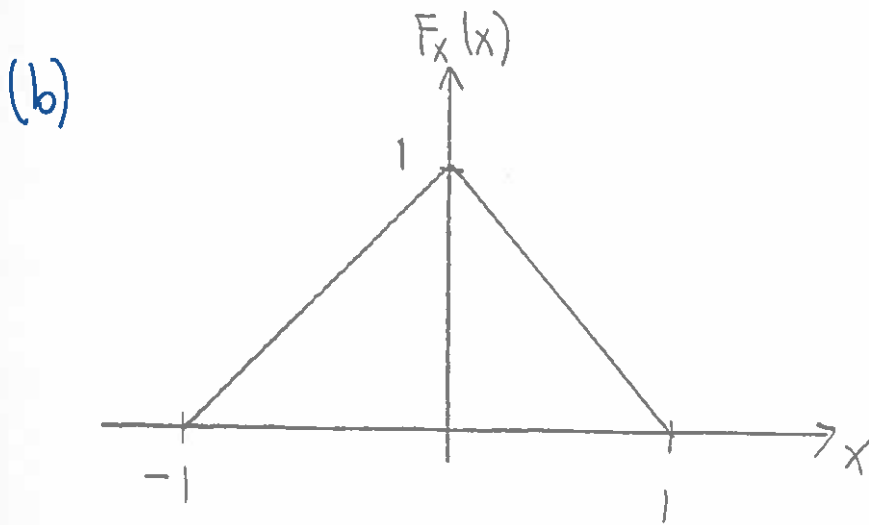
To determine the density function:

$$f_X(x) = \begin{cases} \frac{d}{dx}(x^3) = 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The Image of  $X$  is defined when  $f_X(x) > 0$

$$\text{Im } X = [0, 1]$$

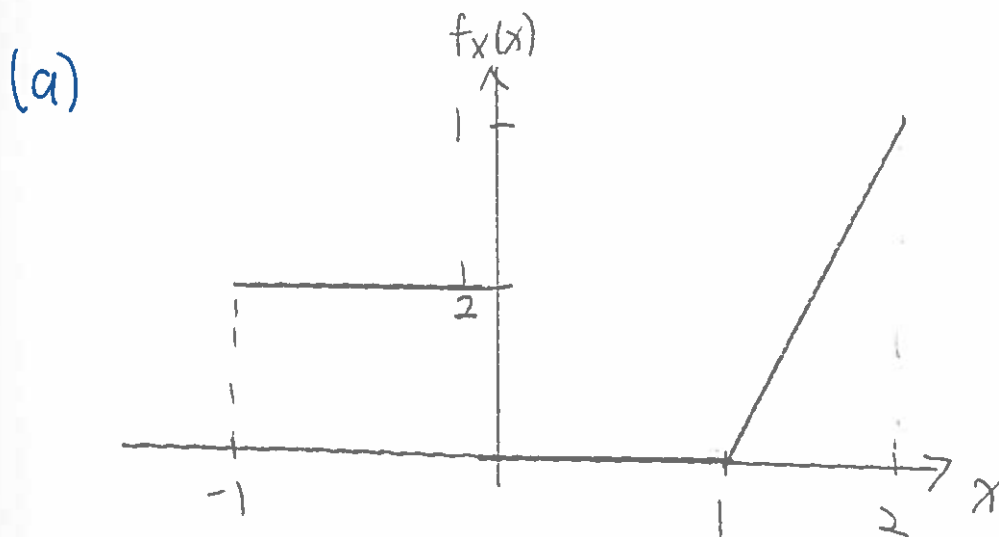
$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \cdot 3x^2 dx = \int_0^1 3x^3 dx \\ &= \left[ \frac{3x^4}{4} \right]_0^1 = \frac{3}{4} \end{aligned}$$



From the graph above, property:  $F_X(x)$  is a non decreasing function is not satisfied.

$\therefore F_X(x)$  is not a distribution function

② The easiest way to check whether a function is a density function ③ is by sketching the graph. To check whether the area under the density function equals 1, it involves integrating the function between the maximum and minimum possible  $x$  values it can take.



Condition (a): Area under the density function  $\left[ \int_{-\infty}^{\infty} f_x(u) du = 1 \right]$

This can be shown by a geometric argument or using a calculus argument.

$$\begin{aligned} \int_{-\infty}^{\infty} f_x(u) du &= \int_{-1}^2 f_x(u) du = \int_{-1}^0 \frac{1}{2} du + \int_0^2 (u-1) du \\ &= \left[ \frac{u}{2} \right]_{-1}^0 + \left[ \frac{u^2}{2} - u \right]_0^2 \\ &= \frac{1}{2} + \left[ \left( \frac{4}{2} - 2 \right) - \left( \frac{1}{2} - 1 \right) \right] \\ &= 1 \end{aligned}$$

Condition (b):  $f_x(x) \geq 0$  for  $x \in \mathbb{R}$

This can be seen straight away from the graph above

Condition (c):  $\text{Im } X = \{x : f_x(u) > 0\}$   
 $= \{[-1, 0] \cup [0, 2]\}$

Therefore  $f_x(x)$  is a density function

(4)

To calculate the cdf, the following formula is used:

$$F_x(x) = P(X \leq x) = \int_{-\infty}^x f_x(u) du$$

The cdf will consist of 5 levels:

$$x \in (-\infty, -1] \quad x \in [-1, 0] \quad x \in [0, 1] \quad x \in [1, 2] \quad x \in [2, \infty)$$

$$\forall x \in (-\infty, -1], F_x(x) = 0$$

$$\forall x \in [-1, 0] \Rightarrow F_x(x) = 0 + \int_{-1}^x \frac{1}{2} du = \left[ \frac{1}{2}u \right]_{-1}^x = \frac{1}{2}x + \frac{1}{2}$$

$$\forall x \in [0, 1] \Rightarrow F_x(x) = \frac{1}{2}$$

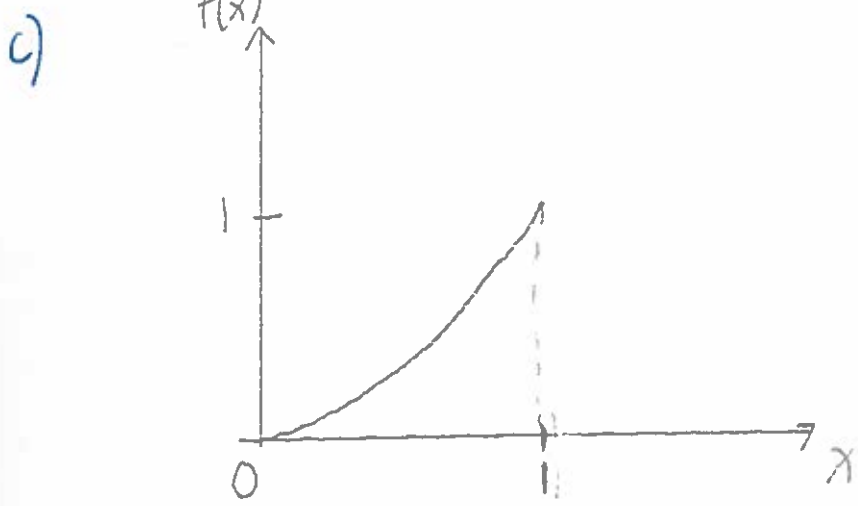
$$\forall x \in [1, 2] \Rightarrow F_x(x) = \frac{1}{2} + \int_1^x (u-1) du = \frac{1}{2} + \left[ \frac{u^2}{2} - u \right]_1^x = 1 + \frac{1}{2}x^2 - x$$

$$\forall x \in [2, \infty) \Rightarrow F_x(x) = 1$$

Putting this all together gives

$$F_x(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}x + \frac{1}{2} & -1 \leq x \leq 0 \\ \frac{1}{2} & 0 \leq x \leq 1 \\ 1 + \frac{1}{2}x^2 - x & 1 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

b)  $f_x(x)$  is not a density function because  $\int_{-2}^2 f_x(x) dx \neq 1$



$$\text{Condition (a)} \Rightarrow \int_0^1 3x^2 dx = [x^3]_0^1 = 1 \quad [\text{satisfied}]$$

$$\text{Condition (b)} \Rightarrow f_x(x) > 0 \text{ for } 0 < x < 1 \quad [\text{satisfied}]$$

$$\begin{aligned} \text{Condition (c)} \Rightarrow \text{Im } X &= \{x : f_x(u) > 0\} \\ &= \frac{1}{3} [0, 1] \quad [\text{satisfied}] \end{aligned}$$

Since all 3 conditions are satisfied,  $f(x)$  is a density function

Next is the cdf

$$\forall x \in (-\infty, 0], F_X(x) = 0$$

$$\forall x \in [0, 1], F_X(x) = \int_0^x 3u^2 du = [u^3]_0^x = x^3$$

$$\forall x \in [1, \infty), F_X(x) = 1$$

Putting this all together gives

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

## Section 2

(6)

1) (a) Using  $\int_{-\infty}^{\infty} f_X(x) dx = 1$ ,

$$\int_0^1 ax^2 dx = 1$$

$$\left[ \frac{ax^3}{3} \right]_0^1 = 1$$

$$\frac{a}{3} = 1$$

$$a = 3$$

(b)  $\forall x \in (-\infty, 0], F_X(x) = 0$

$$\forall x \in [0, 1], F_X(x) = \int_0^x 3u^2 du = \left[ u^3 \right]_0^x = x^3$$

$\forall x \in [1, \infty), F_X(x) = 1$

Putting this all together gives

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$\begin{aligned} \text{(c) } E(X) &= \int_0^1 x \cdot 3x^2 dx = \cancel{\left[ x^3 \right]_0^1} \\ &= \int_0^1 3x^3 dx = \left[ \frac{3}{4} x^4 \right]_0^1 = \frac{3}{4} \end{aligned}$$

$$(d) \quad Y = X^3$$

$$\text{Im} X = [0, 1]$$

$$\text{Im} Y = [0, 1]$$

e) We begin with the distribution function for  $Y$  defined as

$$F_Y(y) = P(Y \leq y)$$

You can rewrite  $Y$  in terms of  $X$  to get the following

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) \\ &= F_X(y^{1/3}) \end{aligned}$$

Substitute  $y^{1/3}$  into the cdf of  $X$  to get:

$$\Rightarrow (y^{1/3})^3 = y$$

The above is defined for  $0 \leq y < 1$

$$\forall y \in \mathbb{R}(-\infty, 0], F_Y(y) = 0$$

$$\forall y \in [1, \infty), F_Y(y) = 1$$

Putting this all together gives:

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$f) \quad E(Y) = \int_0^1 y \, dy = \left[ \frac{y^2}{2} \right]_0^1 = \frac{1}{2}$$

Note: If you cannot solve (e), it is still possible to solve (f) <sup>(8)</sup>

Formula required  $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_x(x) dx$

Applying this formula gives

$$E(Y) = E(X^3) = \int_0^1 3x^2 \cdot x^3 dx = \int_0^1 3x^5 dx$$
$$= \left[ \frac{x^6}{2} \right]_0^1 = \frac{1}{2}$$

Both methods will give the same answer.

### Section 3

1) (a) It is very important to be aware of what values  $x$  can take when working out the limits for the integrals.

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y) dy$$
$$= \int_0^x 8xy dy$$
$$= \left[ 4xy^2 \right]_{y=0}^{y=x} = 4x^3$$

$$f_x(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned}
 (b) f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\
 &= \int_y^1 8xy dx \\
 &= [4x^2y]_{x=y}^{x=1} \\
 &= 4y - 4y^3 = 4y(1-y^2)
 \end{aligned}$$

$$\therefore f_Y(y) = \begin{cases} 4y(1-y^2) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Use the following formula for solving conditional density problems:

$$f_{Y|X}(y|x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)} & f_X(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

For  $f_X(x) > 0$

$$f_{Y|X}(y|x) = \frac{8xy}{4x^3} = \frac{2y}{x^2} \text{ for } 0 < y < x < 1$$

(d) To find  $P(Y > \frac{1}{4} | X = \frac{3}{4})$ ,

You will need to determine  $f_{Y|X}(y|\frac{3}{4}) = \frac{f_{X,Y}(\frac{3}{4}, y)}{f_X(\frac{3}{4})}$

$$f_{Y|X}(y|\frac{3}{4}) = \frac{8 \times \frac{3}{4} \times y}{4 \times (\frac{3}{4})^3} = \frac{6y}{\frac{27}{16}} = \frac{32}{9}y$$

When  $x = \frac{3}{4}$ ,  $y$  can only take values  $0 \leq y \leq \frac{3}{4}$

$$\begin{aligned}
 P\left(Y > \frac{1}{4} \mid X = \frac{3}{4}\right) &= \int_{\frac{1}{4}}^{\frac{3}{4}} f_{Y|X}(y \mid \frac{3}{4}) dy \\
 &= \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{32}{9} y dy = \left[ \frac{32}{18} y^2 \right]_{\frac{1}{4}}^{\frac{3}{4}} \\
 &= \frac{8}{9}
 \end{aligned}$$

$$\begin{aligned}
 (e) E(Y \mid X = \frac{3}{4}) &= \int_0^{\frac{3}{4}} y f_{Y|X}(y \mid \frac{3}{4}) dy \\
 &= \int_0^{\frac{3}{4}} y \cdot \frac{32}{9} y dy \\
 &= \int_0^{\frac{3}{4}} \frac{32}{9} y^2 dy = \left[ \frac{32}{27} y^3 \right]_0^{\frac{3}{4}} = \frac{1}{2}
 \end{aligned}$$

2) (a) The independence property means  $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$f_{X,Y}(x,y) = 3e^{-3x} e^{-y}$$

(b) First find the distribution function

$$F_Z(z) = P(Z \leq z) = P\left(\frac{Y}{X} \leq z\right) = P(Y \leq zX)$$

Then, do a double integral between  $y=0$  and  $y=zX$  and  $x=0$  and  $x=\infty$

$$\begin{aligned}
 F_Z(z) = P(Y \leq zX) &= \int_0^{\infty} \int_0^{y=zX} 3e^{-3x} e^{-y} dy dx \\
 &= \int_0^{\infty} \left[ -3e^{-3x} e^{-y} \right]_0^{zX} dx \\
 &= \int_0^{\infty} -3e^{-3x} (e^{-2x} - 1) dx \\
 &= \left[ \frac{3e^{-(3+2)x}}{2+3} - e^{-3x} \right]_0^{\infty} = 1 - \frac{3}{3+2}
 \end{aligned}$$

To obtain the density function:

(11)

$$f_z(z) = \frac{\partial}{\partial z} \left( 1 - \frac{3}{3+z} \right) = \frac{3}{(3+z)^2}$$

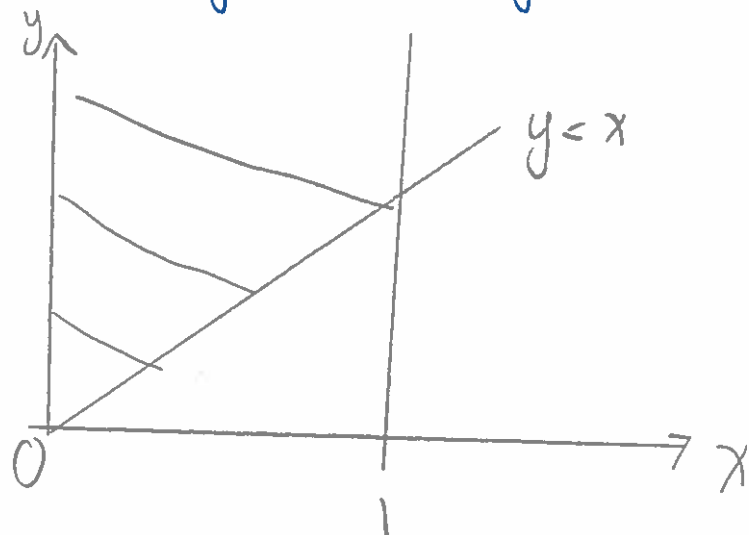
Since  $z = \frac{y}{x}$ ,  $\text{Im } Z = [0, \infty)$

$$f_z(z) = \begin{cases} \frac{3}{(3+z)^2} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} 3(a) \quad f_{x,y}(x,y) &= f_{y|x}(y|x) f_x(x) \\ &= xe^{-xy} \cdot 2x \\ &= 2x^2 e^{-xy} \end{aligned}$$

$$f_{x,y}(x,y) = \begin{cases} 2x^2 e^{-xy} & 0 < x < 1, 0 < y < \infty \\ 0 & \text{aw} \end{cases}$$

To calculate  $P(Y > X)$ , refer to the graph below to determine the shaded region and integral limits



$$\begin{aligned}
 P(Y > X) &= \int_0^1 \int_x^\infty f_{XY}(x, y) dy dx \\
 &= \int_0^1 \int_x^\infty 2x^2 e^{-xy} dy dx \\
 &= \int_0^1 -2xe^{-xy} \Big|_{y=x}^{y=\infty} dx \\
 &= \int_0^1 2xe^{-x^2} dx = 1 - e^{-1}
 \end{aligned}$$

$$(b) f_{Y|X}(y|x) = \begin{cases} xe^{-xy} & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

$f_{Y|X}(y|x)$  has an  $\text{Exp}(x)$  density

$$E(Y|X=\lambda) = \frac{1}{\lambda}$$

#### Section 4

$$(1) (a) M_{\bar{X}}(\theta) = E(e^{\bar{X}\theta}) = E\left(e^{\theta \frac{(X_1 + X_2 + \dots + X_n)}{n}}\right)$$

~~iid property~~

independence  $E\left(e^{\frac{\theta X_1}{n}}\right) E\left(e^{\frac{\theta X_2}{n}}\right) \dots E\left(e^{\frac{\theta X_n}{n}}\right)$

$X$ 's are

identically

distributed

$$\left[E\left(e^{\frac{\theta X}{n}}\right)\right]^n = \left(M_X\left(\frac{\theta}{n}\right)\right)^n$$

(b) You will need the mgf of a Gamma distribution

$$\text{mgf of Gamma } \Gamma(t, \lambda) = \left(\frac{\lambda}{\lambda - \theta}\right)^t$$

$$M_{\bar{X}}(\theta) = \left( M_X \left( \frac{\theta}{n} \right) \right)^n = \left[ \left( \frac{3}{3n - \frac{\theta}{n}} \right)^{n-2} \right]^n$$

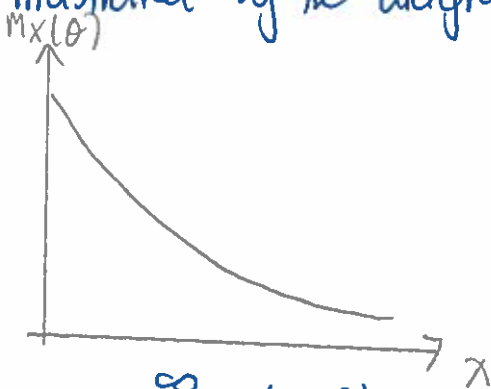
$$= \left( \frac{3}{3n - \frac{\theta}{n}} \right)^{2n} = \left( \frac{3n}{3n - \theta} \right)^{2n}$$

By the uniqueness of the mgf,  $\bar{X} \sim \Gamma(2n, 3n)$

$$2) (a) M_X(\theta) = E(e^{\theta x}) = \int_0^{\infty} e^{\theta x} \lambda e^{-\lambda x} dx \quad [\text{density of exponential}(\lambda)]$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - \theta)x} dx$$

For this integral to be finite, we need  $\lambda - \theta > 0$ . This is illustrated by the diagram below



$$\Rightarrow \lambda \int_0^{\infty} e^{-(\lambda - \theta)x} dx = \lambda \left[ \frac{-e^{-(\lambda - \theta)x}}{\lambda - \theta} \right]_0^{\infty}$$

$$= \frac{\lambda}{\lambda - \theta} \quad \text{if } \theta < \lambda$$

$M_X(\theta)$  is undefined for cases  $\lambda - \theta < 0$  and  $\lambda - \theta = 0$ , because the integral is infinite.

$$(b) M_Y(\theta) = E(e^{\theta Y}) = E(e^{c\theta X}) = M_X(c\theta)$$

$$M_Y(\theta) = \frac{\lambda}{\lambda - c\theta} = \frac{\lambda/c}{\frac{\lambda}{c} - \theta}$$

This is the mgf of  $\text{Exp}\left(\frac{\lambda}{c}\right)$

By the uniqueness of the mgf,  $Y \sim \text{Exp}\left(\frac{\lambda}{c}\right)$

### Section 5

$$(1) (a) (i) X_i \sim \text{Exp}(3)$$

By definition

$$M_T(\theta) = E(e^{\theta T}) = E(e^{\theta(X_1 + \dots + X_{100})})$$

$$\stackrel{\text{independence}}{=} E(e^{\theta X_1}) E(e^{\theta X_2}) \dots E(e^{\theta X_{100}})$$

The  $X$ 's are identically distributed so,

$$= [E(e^{\theta X_1})]^n = \left(\frac{\lambda}{\lambda - \theta}\right)^n$$

By the uniqueness of MGF's,  $T_{100} \sim \Gamma(n, \lambda)$   
 $\Rightarrow T_{100} \sim \Gamma(100, 3)$

(ii) Suppose  $X_1, \dots, X_n$  are iid RV's with  $E(X_i) = \mu$  and  $\text{Var}(X_i) = \sigma^2$   
 and let  $S_n = X_1 + \dots + X_n$

Then for any  $a < b$

$$P\left(a \leq \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq b\right) \Rightarrow P(a \leq Z \leq b) \text{ as } n \rightarrow \infty$$

where  $Z \sim N(0, 1)$

$$P(T_{100} \leq 35) = P\left(\frac{T_{100} - n\mu}{\sqrt{\sigma^2 n}} \leq \frac{35 - n\mu}{\sqrt{\sigma^2 n}}\right)$$

$$n\mu = 100E(X_i) = \frac{100}{3}$$

$$n\sigma^2 = 100 \times \text{Var}(X_i) = \frac{100}{9}$$

By the CLT, we get:

$$\approx P\left(z \leq \frac{35 - \frac{100}{3}}{\sqrt{\frac{100}{9}}}\right) = P\left(z \leq \frac{1}{2}\right) = 0.6915$$

$$Z \sim N(0, 1)$$

(b) Note that  $Y \sim \text{Bin}(150, 0.4)$ .

To find  $P(Y \geq 50)$ , apply the CLT

$$P(Y \geq 50) = P(Y \geq 49.5) \leftarrow \text{Continuity correction is applied here because } Y \text{ is a discrete distribution}$$

$$E(Y) = np = 60 \quad \text{Var}(Y) = np(1-p) = 36$$

$$\Rightarrow P\left(\frac{Y - 60}{6} \geq \frac{49.5 - 60}{6}\right) \stackrel{\text{CLT}}{\approx} P\left(z \geq -\frac{7}{4}\right) = P\left(z \leq \frac{7}{4}\right) \approx 0.95$$

2(a) Repeat the CLT definition mentioned in 1(a)(ii)

Let  $T = X_1 + \dots + X_{50}$  (time to process 50 claims)

$$P(T \leq 120) = P(X_1 + \dots + X_{50} \leq 120)$$

$$= P\left(\frac{X_1 + \dots + X_{50} - 125}{\sqrt{62.5}} \leq \frac{120 - 125}{\sqrt{62.5}}\right)$$

$$\stackrel{\text{CLT}}{\approx} P(Z \leq -0.63) = 0.2643$$

$$E(T) = 50E(X) = 50 \times \frac{5}{2} = 125$$

$$\text{Var}(T) = 50\text{Var}(X) = 62.5 \times 50 \left(\frac{5}{12}\right) = 62.5$$

$$(b) Y \sim \text{Bin}(50, 0.3)$$

(16)

$$P(Y \geq 20) = P(Y \geq 20.5) \quad [\text{Use continuity correction as in 1(b)}]$$

$$= P\left(\frac{Y-15}{\sqrt{10.5}} \geq \frac{20.5-15}{\sqrt{10.5}}\right)$$

$$E(Y) = 50(0.3) = 15 \quad \text{Var}(Y) = 50 \times 0.3 \times 0.7 = 10.5$$

$$\stackrel{\text{CLT}}{\approx} P(Z \geq 1.70) = 0.0446$$