Time Series Revision Class Solutions

1(a)(i) $Tx_t = \frac{1}{3}[1,1,1]x_t$

$$= \frac{1}{3}(x_{t-1} + x_t + x_{t+1}) = \frac{1}{3}x_{t-1} + \frac{1}{3}x_t + \frac{1}{3}x_{t+1}$$

(ii) $y_t = T^2 x_t$

$$= T(Tx_t)$$

$$= T\left(\tfrac{1}{3}\right)(x_{t-1} + x_t + x_{t+1})$$

$$= \frac{1}{3} \times \frac{1}{3}\left[(x_{t-2} + x_{t-1} + x_t) + (x_{t-1} + x_t + x_{t+1}) + (x_t + x_{t+1} + x_{t+2})\right]$$

$$= \frac{1}{9}(x_{t-2} + 2x_{t-1} + 3x_t + 2x_{t+1} + x_{t+2})$$

Since y_t depends on $x_{t-2}, x_{t-1}, x_t, x_{t+1}, x_{t+2}$

$\Rightarrow$ It uses $\underline{\underline{5}}$ data points of the original series $\{x_t\}$.

(b) < This just comes straight from the notes. >

A process $\{Y_t\}$ has "second-order stationarity" (i.e. weak stationarity) if $E(Y_t)$, $Var(Y_t)$ and $Cov(Y_t, Y_{t+k})$ all exist, are finite, and do not depend on t.

In other words, $E(Y_t) = \mu$

$$Var(Y_t) = \sigma^2$$

$$Cov(Y_t, Y_{t+k}) = \gamma_k$$

(c)(i) $Y_t = TY_{t-2} + 3TZ_{t-1}$

$$= \frac{1}{3}[1,1,1]Y_{t-2} + 3\left(\tfrac{1}{3}\right)[1,1,1]Z_{t-1}$$

$$= \frac{1}{3}(Y_{t-3} + Y_{t-2} + Y_{t-1}) + (Z_{t-2} + Z_{t-1} + Z_t)$$

Rearranging we get:

$$Y_t - \frac{1}{3}Y_{t-1} - \frac{1}{3}Y_{t-2} - \frac{1}{3}Y_{t-3} = Z_t + Z_{t-1} + Z_{t-2}$$

$$\left(1 - \frac{1}{3}B - \frac{1}{3}B^2 - \frac{1}{3}B^3\right)Y_t = (1 + B + B^2)Z_t$$

$$\Rightarrow \phi(B) = 1 - \frac{1}{3}B - \frac{1}{3}B^2 - \frac{1}{3}B^3$$

$$\theta(B) = 1 + B + B^2$$

(ii) For $\{Y_t\}$ to be stationary, we need the roots of $\phi(B)$ to be outside the unit circle. <i.e. roots of $\phi(\lambda)$ must have modulus > 1.>

$\phi(B) = 1 - \frac{1}{3}B - \frac{1}{3}B^2 - \frac{1}{3}B^3$ (from part (c)(i)).

$\phi(\lambda) = 1 - \frac{1}{3}\lambda - \frac{1}{3}\lambda^2 - \frac{1}{3}\lambda^3$

<The associated λ polynomial is given as $\lambda^3 \phi(\frac{1}{\lambda})$>.

$\Rightarrow \lambda^3 - \frac{1}{3}\lambda^2 - \frac{1}{3}\lambda - \frac{1}{3} = 0$

$\times 3)$ $\quad 3\lambda^3 - \lambda^2 - \lambda - 1 = 0$

$(\lambda - 1)(3\lambda^2 + 2\lambda + 1) = 0$

$\lambda - 1 = 0 \qquad\qquad 3\lambda^2 + 2\lambda + 1 = 0$

One of the roots, $\lambda = 1$, which is not strictly larger than 1.

$\Rightarrow \{Y_t\}$ is not stationary.

(d)(i) $X_t = \theta(B) Z_t = 3 T Z_{t-1}$

$= Z_{t-2} + Z_{t-1} + Z_t$

$\gamma_k = Cov(X_t, X_{t-k})$

$= E(X_t X_{t-k}) - E(X_t)E(X_{t-k})$

Note: $E(X_t) = E(Z_{t-2} + Z_{t-1} + Z_t)$

$= E(Z_{t-2}) + E(Z_{t-1}) + E(Z_t)$

$= 0$

$\therefore \gamma_k = E(X_t X_{t-k})$

$= E\left((Z_{t-2} + Z_{t-1} + Z_t)(Z_{t-k-2} + Z_{t-k-1} + Z_{t-k})\right)$

<To work this out, remember that for a white noise process,

$Cov(Z_t, Z_s) = E(Z_t, Z_s) = \begin{cases} \sigma_Z^2 & \text{if } t = s \\ 0 & \text{otherwise.} \end{cases}$ (*) >

When $k = 0$:

$\gamma_0 = E((Z_{t-2} + Z_{t-1} + Z_t)(Z_{t-2} + Z_{t-1} + Z_t))$

$= E(Z_{t-2}^2) + E(Z_{t-1}^2) + E(Z_t^2)$ <all other terms are 0 due to (*)>

$$\Rightarrow \gamma_0 = \sigma_Z^2 + \sigma_Z^2 + \sigma_Z^2$$
$$= 1+1+1$$
$$= 3$$

When $k=1$:

$$\gamma_1 = E((Z_{t-2}+Z_{t-1}+Z_t)(Z_{t-3}+Z_{t-2}+Z_{t-1}))$$
$$= E(Z_{t-2}^2) + E(Z_{t-1}^2)$$
$$= 1+1$$
$$= 2$$

When $k=2$:

$$\gamma_2 = E((Z_{t-2}+Z_{t-1}+Z_t)(Z_{t-4}+Z_{t-3}+Z_{t-2}))$$
$$= E(Z_{t-2}^2)$$
$$= 1$$

When $k=3$:

$$\gamma_k = E((Z_{t-2}+Z_{t-1}+Z_t)(Z_{t-5}+Z_{t-4}+Z_{t-3}))$$
$$= 0$$

This is true for all $k \geqslant 3$.

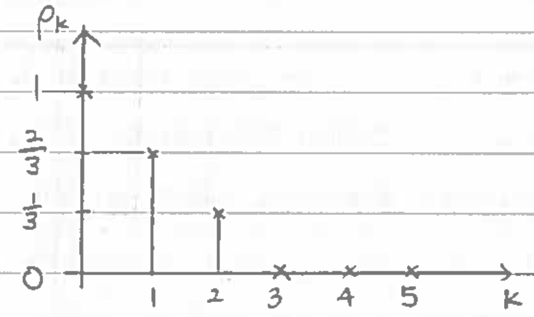
Putting all this together, we have:

$k=0$; $\gamma_k = 3$; $\rho_k = 1$

$k=1$; $\gamma_k = 2$; $\rho_k = \frac{2}{3}$ $\quad \langle \rho_k = \frac{\gamma_k}{\gamma_0} \rangle$

$k=2$; $\gamma_k = 1$; $\rho_k = \frac{1}{3}$

$k \geqslant 3$; $\gamma_k = 0$; $\rho_k = 0$



<Note: Not necessary to draw crosses – I just did it so it is easier to align.>

(ii) Two ways to do this:

(A) Do it directly, i.e. $f^*(\omega) = G(e^{-i\omega})$

where $G(z)$ is the autocorrelation generating function,

i.e. $G(z) = 1 + \sum_{k=1}^{\infty} \rho_k (z^k + z^{-k})$

Note: $e^{-ik\omega} + e^{ik\omega} = (\cos k\omega - i \sin k\omega) + (\cos k\omega + i \sin k\omega)$

$= 2 \cos k\omega$.

$$\Rightarrow f^*(\omega) = 1 + \sum_{k=1}^{\infty} \rho_k (2 \cos k\omega)$$
$$= 1 + \rho_1 (2\cos\omega) + \rho_2 (2 \cos 2\omega) + \rho_3 (2\cos 3\omega) + \dots$$
$$= 1 + \tfrac{2}{3}(2\cos\omega) + \tfrac{1}{3}(2\cos 2\omega) + 0$$
$$= 1 + \tfrac{4}{3}\cos\omega + \tfrac{2}{3}\cos 2\omega \;_{\#}$$

(B) Use "The Lemma", i.e. $f_y^*(\omega) = |a(\omega)|^2 f_x^*(\omega) \frac{\sigma_x^2}{\sigma_y^2}$.

Remember, this is used when $Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j} = A(B) X_t$,

and $a(\omega) = A(e^{-i\omega})$.

In this case, we want $f^*(\omega) = f_x^*(\omega) = |a(\omega)|^2 f_z^*(\omega) \frac{\sigma_z^2}{\sigma_x^2}$.

Since $X_t = Z_{t-2} + Z_{t-1} + Z_t$, we have $A(B) = B^2 + B + 1$.

$$a(\omega) = A(e^{-i\omega})$$
$$= (e^{-i\omega})^2 + e^{-i\omega} + 1$$
$$= e^{-2i\omega} + e^{-i\omega} + 1$$
$$= (\cos 2\omega - i \sin 2\omega) + (\cos\omega - i\sin\omega) + 1$$
$$= \cos 2\omega + \cos\omega + 1 - i(\sin 2\omega + \sin\omega)$$

$$|a(\omega)| = \sqrt{(\cos 2\omega + \cos\omega + 1)^2 + (\sin 2\omega + \sin\omega)^2}$$
$$= \big[\cos^2 2\omega + \cos^2\omega + 1 + 2(\cos 2\omega \cos\omega + \cos 2\omega + \cos\omega)$$
$$+ \sin^2 2\omega + 2\sin 2\omega \sin\omega + \sin^2\omega\big]^{\frac{1}{2}}$$
$$= \big[(\cos^2 2\omega + \sin^2 2\omega) + (\cos^2\omega + \sin^2\omega) + 1$$
$$+ 2(\cos 2\omega\cos\omega + \sin 2\omega \sin\omega) + 2(\cos 2\omega + \cos\omega)\big]^{\frac{1}{2}}$$
$$= \big[1 + 1 + 1 + 2\cos(2\omega - \omega) + 2(\cos 2\omega + \cos\omega)\big]^{\frac{1}{2}}$$
$$= (3 + 2\cos\omega + 2\cos 2\omega + 2\cos\omega)^{\frac{1}{2}}$$

$$|a(w)| = (3 + 4\cos w + 2\cos 2w)^{\frac{1}{2}}$$
$$\Rightarrow |a(w)|^2 = 3 + 4\cos w + 2\cos 2w$$

Note: This uses trigonometric formulae such as

$$\sin^2 x + \cos^2 x = 1$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

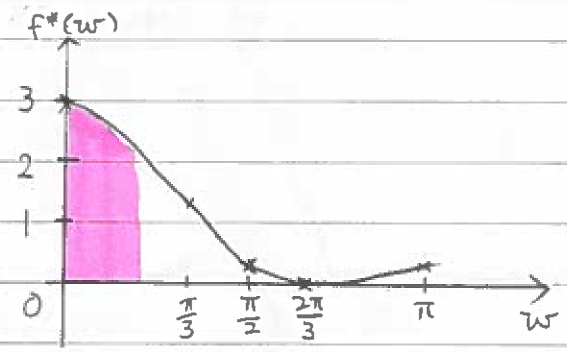
$f_Z^*(w) = 1$. Since Z_t is a white noise process, all $\rho_k = 0$ for $k \neq 0$.
$\sigma_Z^2 = 1$ from the question.
$\sigma_X^2 = \gamma_0 = 3$ (from part (d)(i)).

$$\Rightarrow f^*(w) = (3 + 4\cos w + 2\cos 2w)(1)\left(\tfrac{1}{3}\right)$$
$$= 1 + \tfrac{4}{3}\cos w + \tfrac{2}{3}\cos 2w$$

As you can see, A was much more straightforward. B was a bigger hassle and required trigonometric identities. However, it would still be important to be able to do B, in case the exam specifically asks for it to be solved using The Lemma.

(iii) To sketch this, it would be helpful to work out $f^*(w)$ for some key values of w.

$w = 0 \Rightarrow f^*(w) = 3$
$w = \frac{\pi}{3} \Rightarrow f^*(w) = \frac{4}{3}$
$w = \frac{\pi}{2} \Rightarrow f^*(w) = \frac{1}{3}$
$w = \frac{2\pi}{3} \Rightarrow f^*(w) = 0$
$w = \pi \Rightarrow f^*(w) = \frac{1}{3}$



There is a high weight on low frequencies (highlighted area) i.e. long periods due to the smoothing effect the MA of length 3 has on the white noise.

2(a) <This is asking for the proof of The Lemma, which is directly in the notes.>

$$\gamma_k^Y = Cov(Y_t, Y_{t+k})$$
$$= Cov\left(\sum_{j=-\infty}^{\infty} a_j X_{t-j}, \sum_{i=-\infty}^{\infty} a_i X_{t+k-i}\right)$$
$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_i a_j \, Cov(X_{t-j}, X_{t+k-i})$$
$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_i a_j \, \gamma_{k-i+j}^X$$

$$C_y(z) = \sum_{k=-\infty}^{\infty} \gamma_k^Y z^k$$
$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_i a_j \gamma_{k-i+j}^X z^k$$
$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_i z^i a_j z^{-j} \gamma_{k-i+j}^X z^{k-i+j}$$
$$= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_i z^i a_j z^{-j} \gamma_k^X z^k$$
$$= \left(\sum_{i=-\infty}^{\infty} a_i z^i\right)\left(\sum_{j=-\infty}^{\infty} a_j z^{-j}\right)\left(\sum_{k=-\infty}^{\infty} \gamma_k^X z^k\right)$$
$$= A(z)A(z^{-1})C_x(z) \; \#$$

(b)(i) $V_t = \alpha V_{t-1} + Z_t$

$$E(V_t) = E(\alpha V_{t-1} + Z_t)$$
$$= \alpha E(V_{t-1}) + E(Z_t)$$
$$= \alpha E(V_{t-1})$$

If we want V_t to be stationary, $E(V_t) = 0$.

$$\sigma_v^2 = E(V_t^2) - (E(V_t))^2$$
$$= E((\alpha V_{t-1} + Z_t)(\alpha V_{t-1} + Z_t))$$
$$= E(\alpha^2 V_{t-1}^2 + 2\alpha V_{t-1} Z_t + Z_t^2)$$
$$= \alpha^2 E(V_{t-1}^2) + E(Z_t^2)$$
$$= \alpha^2 \sigma_v^2 + \sigma_z^2$$

$$(1-\alpha^2)\sigma_v^2 = \sigma_z^2$$
$$\Rightarrow \sigma_v^2 = \frac{\sigma_z^2}{1-\alpha^2} \; \#$$

$$Y_t = V_t + W_t$$
$$E(Y_t) = E(V_t + W_t)$$
$$= E(V_t) + E(W_t)$$
$$= 0.$$

$$\sigma_Y^2 = E(Y_t^2) - (E(Y_t))^2$$
$$= E((V_t + W_t)(V_t + W_t))$$
$$= E(V_t^2 + 2V_t W_t + W_t^2)$$
$$= E(V_t^2) + 2E(V_t W_t) + E(W_t^2)$$

Note: V_t only depends on Z_t, and W_t is independent of Z_t, so $E(V_t W_t) = 0$.

$$\sigma_Y^2 = E(V_t^2) + E(W_t^2)$$
$$= \sigma_V^2 + \sigma_W^2$$
$$= \frac{\sigma_Z^2}{1-\alpha^2} + \sigma_W^2$$

If we want everything in terms of σ_W^2, use $\lambda = \frac{\sigma_Z^2}{\sigma_W^2} \Rightarrow \sigma_Z^2 = \lambda \sigma_W^2$.

$$\sigma_Y^2 = \frac{\lambda \sigma_W^2}{1-\alpha^2} + \sigma_W^2$$
$$= \left(\frac{\lambda}{1-\alpha^2} + 1\right)\sigma_W^2$$

(ii) Here, it would be easier to use The Lemma instead of working from first principles.

First we note that since V_t and W_t are independent,
$$C_Y(z) = C_V(z) + C_W(z).$$
This is because:
$$C_Y(z) = \sum_{k=-\infty}^{\infty} \gamma_k^Y$$
$$= \sum_{k=-\infty}^{\infty} Cov(Y_t, Y_{t+k})$$
$$= \sum_{k=-\infty}^{\infty} Cov((V_t + W_t), (V_{t+k} + W_{t+k}))$$
$$= \sum_{k=-\infty}^{\infty} E((V_t + W_t)(V_{t+k} + W_{t+k})) - \underbrace{E(Y_t)E(Y_{t+k})}_{=0}$$
$$= \sum_{k=-\infty}^{\infty} [E(V_t V_{t+k}) + E(W_t W_{t+k})] \text{ because all other terms } = 0$$
$$= \sum_{k=-\infty}^{\infty} (\gamma_k^V + \gamma_k^W) = C_V(z) + C_W(z).$$

$$V_t = \alpha V_{t-1} + Z_t$$
$$(1-\alpha B) V_t = Z_t$$
$$V_t = (1-\alpha B)^{-1} Z_t$$
$$\Rightarrow A(z) = (1-\alpha z)^{-1}$$

The Lemma then gives:
$$C_v(z) = A(z) A(z^{-1}) C_z(z)$$
$$= (1-\alpha z)^{-1}(1-\tfrac{\alpha}{z})^{-1}\sigma_z^2$$

$$C_w(z) = \sigma_w^2$$

$$\Rightarrow C_y(z) = (1-\alpha z)^{-1}(1-\tfrac{\alpha}{z})^{-1}\sigma_z^2 + \sigma_w^2$$
$$= (1-\alpha z)^{-1}(1-\tfrac{\alpha}{z})^{-1}(\lambda\sigma_w^2) + \sigma_w^2$$
$$= [1 + \lambda(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}]\sigma_w^2$$

$$G_y(z) = \frac{C_y(z)}{\sigma_y^2}$$

$$\Rightarrow G_y(z) = \frac{\cancel{(1+\lambda(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1})\sigma_w^2}}{(\frac{\lambda}{1-\alpha^2}+1)\cancel{\sigma_w^2}}$$

$$= \frac{1+\lambda(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}}{1+\lambda(1-\alpha^2)^{-1}}$$

*

(iii) From (b)(i):
$$\sigma_v^2 = \frac{\sigma_z^2}{1-\alpha} \qquad \sigma_y^2 = \sigma_v^2 + \sigma_w^2$$

If $\sigma_z^2 = 0$, $\sigma_v^2 = 0$. This means V_t is a constant with no variation.

If $\sigma_z^2 = 0$, $\sigma_y^2 = 0 + \sigma_w^2 = \sigma_w^2$. Also note: $Y_t = V_t + W_t$. V_t is a constant and W_t is noise, so Y_t becomes noise as well.

(iv)(I) From (ii), $C_v(z) = (1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}\sigma_z^2$

$$G_v(z) = \frac{C_v(z)}{\sigma_v^2}$$

$$= \frac{(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}\cancel{\sigma_z^2}}{(1-\alpha^2)^{-1}\cancel{\sigma_z^2}}$$

$$= \frac{(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}}{(1-\alpha^2)^{-1}}$$

$$= \frac{1-\alpha^2}{(1-\alpha z)(1-\alpha z^{-1})}$$

$$= \frac{1-\alpha^2}{1+\alpha^2-\alpha(z+z^{-1})}$$

$$f_v^*(w) = G_v(e^{-iw})$$

$$= \frac{1-\alpha^2}{1+\alpha^2-\alpha(2\cos w)}$$

When $\alpha = -0.7$, $f_v^*(w) = \frac{1-(-0.7)^2}{1+(-0.7)^2-(-0.7)(2\cos w)}$

$$= \frac{0.51}{1.49+1.4\cos w}$$

From (ii), $G_y(z) = \frac{1+\lambda(1-\alpha z)^{-1}(1-\alpha z^{-1})^{-1}}{1+\lambda(1-\alpha^2)^{-1}}$

$$= \frac{1+\frac{\lambda}{(1-\alpha z)(1-\alpha z^{-1})}}{1+\frac{\lambda}{1-\alpha^2}}$$

$$= \frac{1+\frac{\lambda}{1+\alpha^2-\alpha(z+z^{-1})}}{1+\frac{\lambda}{1-\alpha^2}}$$

$$f_y^*(w) = G_y(e^{-iw})$$

$$= \frac{1+\frac{\lambda}{1+\alpha^2-\alpha(2\cos w)}}{1+\frac{\lambda}{1-\alpha^2}}$$

When $\alpha = -0.7$, $f_y^*(w) = \frac{1+\frac{\lambda}{1+(-0.7)^2-(-0.7)(2\cos w)}}{1+\frac{\lambda}{1-(-0.7)^2}} = \frac{1+\frac{\lambda}{1.49+1.4\cos w}}{1+\frac{\lambda}{0.51}}$

(2) First graph shows $f^*_v(\omega)$.

→ The area under the graph is highest for higher values of ω

⇒ (higher frequencies i.e. low periods are more important. This is due to $\alpha = -0.7$.)

Second and third graphs show $f^*_y(\omega)$, with $\lambda = 1$ and $\lambda = 0.2$ respectively.

→ Remember: $\lambda = \frac{\sigma_z^2}{\sigma_w^2}$, i.e. "signal to noise" ratio.

→ Also, W_t is noise so $f^*_w(\omega) = 1$.

→ $f^*_y(\omega)$ is sort of like a "weighted average" of $f^*_v(\omega)$ and $f^*_w(\omega)$; look at the formula of $f^*_y(\omega)$ from part (1) and compare it with $f^*_v(\omega)$.

Second graph ($\lambda = 1$):

→ (The added noise W_t has little effect on the frequency)

⇒ higher frequencies still dominate.

→ This makes sense intuitively; signal to noise ratio is high, so the effect of the signal is strong.

⇒ $f^*_y(\omega)$ resembles $f^*_v(\omega)$.

Third graph ($\lambda = 0.2$):

→ (The added noise W_t is much stronger, bringing the spectrum closer to that of noise.)

⇒ higher frequencies no longer dominate as much.

→ Again, this makes sense intuitively; signal to noise ratio is low, so the effect of the noise is stronger.

⇒ $f^*_y(\omega)$ resembles $f^*_w(\omega)$ which is a horizontal line at 1.

< Marks are given to sentences bracketed in red. >