F70TS2: Time Series

Exam Solutions 2014

1. a) (i) For a time series process $\{Y_t\}$, the joint distribution of $(Y_{t+1}, \ldots, Y_{t+k})$ does not depend on t for any $k = 1, 2, 3 \ldots$. [2]

(ii) A sequence of iid random variables $Z_1, Z_2, \ldots$, for example. [2]

(iii) For a process $\{Y_t\}$, the mean $\mathbb{E}[Y_t]$ and variance $\text{Var}(Y_t)$ exist and do not depend on t. The covariances $\text{Cov}(Y_s, Y_t)$ exist and only depend on the difference $t - s$. [2]

b) (i) We find the coefficient a_0 in $f(t)$ by minimising [1]

$$S = \sum_{t=-2}^{2} \left(x_t - a_0 - a_1 t - a_2 t^2 - a_3 t^3\right)^2$$

Now,

$$\frac{\partial S}{\partial a_0} = -2 \sum_{t=-2}^{2} \left(x_t - a_0 - a_1 t - a_2 t^2 - a_3 t^3\right) = 10a_0 + 20a_2 - 2\sum_{t=-2}^{2} x_t \tag{1}$$

$$\frac{\partial S}{\partial a_2} = -2 \sum_{t=-2}^{2} t^2 \left(x_t - a_0 - a_1 t - a_2 t^2 - a_3 t^3\right) = 20a_0 + 68a_2 - 2\sum_{t=-2}^{2} t^2 x_t \tag{2}$$

Set both of these to zero and solve (**??**) and (**??**) simultaneously to get [2]

$$\frac{70}{17} a_o = 2\sum_{t=-2}^{2} x_t - \frac{10}{17} \sum_{t=-2}^{2} t^2 x_2$$

and hence

$$a_0 = \frac{1}{35}(-3x_{-2} + 12x_{-1} + 17x_0 + 12x_1 - 3x_2)$$

Hence, $T = \frac{1}{35}[-3, 12, 17, 12, -3]$, as required. [2]

(ii) [1]

$$\text{Var}(TZ_t) = \left(\frac{3^2 + 12^2 + 17^2 + 12^2 + 3^2}{35^2}\right)\sigma^2 = \left(\frac{17}{35}\right)\sigma^2$$

[1]

c) (i)

$$Y_t = \frac{1}{35}\left(-3Y_{t-5} + 12Y_{t-4} + 17Y_{t-3} + 12Y_{t-2} - 3Y_{t-1}\right) - \frac{2}{35}\left(-3Z_{t-4} + 12Z_{t-3} + 17Z_{t-2} + 12Z_{t-1} - 3Z_t\right)$$

Hence,

$$\frac{1}{35}\left(35 + 3B - 12B^2 - 17B^3 - 12B^4 + 3B^5\right) Y_t = \frac{2}{35}\left(3 - 12B - 17B^2 - 12B^3 + 3B^4\right) Z_t$$

We thus have

$$\phi(B) = \frac{1}{35}\left(35 + 3B - 12B^2 - 17B^3 - 12B^4 + 3B^5\right)$$

and [2]

$$\theta(B) = \frac{2}{35}\left(3 - 12B - 17B^2 - 12B^3 + 3B^4\right)$$

[2]

(ii) $\phi(B)$ has all its roots outside the unit circle. [2]

(iii) Using the hint, $\phi(z) = 0$ has a solution $z = 1$, so $\{Y_t\}$ is not stationary. [1]

[Total Q1: 20]

2. a) (i) The autocorrelation function ρ_k is non–zero for lags $k = 0, 1, \ldots, q$, and zero for lags $k > q$ (i.e. it cuts off after lag q). [2]

The partial autocorrelation function tails off as the lag increases. [1]

(ii) For ACF 1, an MA model seems appropriate (the ACF seems to cut off) with $q = 4$. [2]

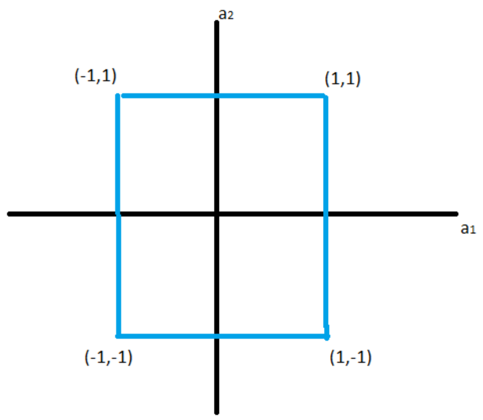
For ACF 2, an MA model does not seem appropriate. Arguably, the autocorrelation cuts off after lag 10, but the principle of parsimony suggests we would be better fitting a model with fewer than 11 parameters. [1]

b) (i) All zeros of $1 + \beta_1 z + \beta_2 z^2$ lie outside the unit circle. [1]

(ii) Write $1 + \beta_1 z + \beta_2 z^2 = (1 + a_1 z)(1 + a_2 z)$, where

$$a_1 + a_2 = \beta_1 \qquad \text{and} \qquad a_1 a_2 = \beta_2 \qquad (3)$$

For the solutions to $(1 + a_1 z)(1 + a_2 z) = 0$ to lie outside the unit circle, we [1] need $|1/a_i| > 1$, i.e. $|a_i| < 1$ for $i = 1, 2$ (i.e. points within the square on the following sketch). [1]



[1]

Mapping this into the (β_1, β_2) plane, this square becomes the triangle defined by the three inequalities given. To see this, note that (by (**??**))

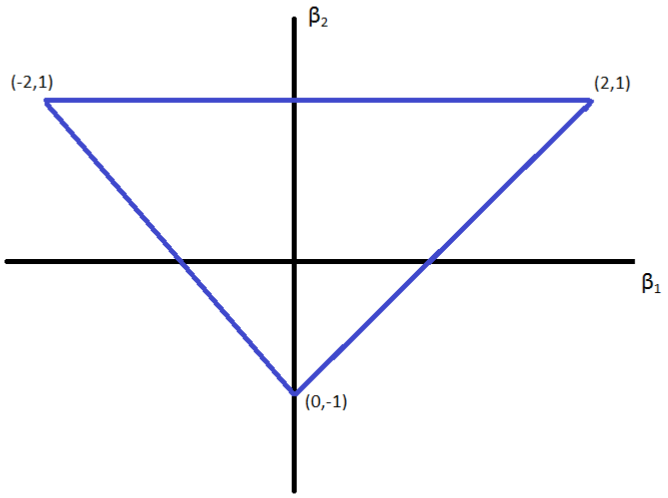
$$a_1 = \beta_1 - \frac{\beta_2}{a_1}$$

and so the line $a_1 = 1$ maps to the line $\beta_2 - \beta_1 = -1$. Similarly, the line $a_1 = -1$ maps to the line $\beta_2 + \beta_1 = -1$.

Finally, we also have that

$$|\beta_2| = |a_1||a_2| < 1$$

Thus, we have the three inequalities given, which define the triangular region shown in the sketch below. [2]



[2]

c) For the MA process defined by $Y_t = \theta(B)Z_t$ to be invertible, we need all roots of $\theta(B)$ to lie outside the unit circle.

(i) $\theta(B) = 1 + 0.8B$, so the root is $-1/0.8 = -1.25$. This is outside the unit circle, so the process is invertible. [2]

(ii) $\theta(B) = 1+0.8B-0.3B^2$. This violates the first inequality of part b(ii) above, so the process is not invertible. [2]

(iii) $\theta(B) = 1 - 3B + 3B^2 - B^3 = (1 - B)^3$. All roots lie on the unit circle, so the process is not invertible. [2]

[Total Q2: 20]

3. a) $p = q = 1$ and $d = 0$ [1]

b) $Y_t = 0.7Y_{t-1} + Z_t + 0.2Z_{t-1}$. Using this defining equation, we have that by

- taking the covariance with Z_t,

$$\text{Cov}(Y_t, Z_t) = \sigma^2 \tag{4}$$

- taking the covariance with Z_{t-1},

$$\begin{aligned}\text{Cov}(Y_t, Z_{t-1}) &= 0.7\text{Cov}(Y_{t-1}, Z_{t-1}) + \text{Cov}(Z_t, Z_{t-1}) \\ &\quad +0.2\text{Cov}(Z_{t-1}, Z_{t-1}) \\ &= 0.7\sigma^2 + 0.2\sigma^2 \qquad \text{by } (\textbf{??}) \\ &= 0.9\sigma^2 \end{aligned} \tag{5}$$

- taking the covariance with Y_t,

$$\begin{aligned}\gamma_0 = \text{Cov}(Y_t, Y_t) &= 0.7\text{Cov}(Y_{t-1}, Y_t) + \text{Cov}(Z_t, Y_t) + 0.2\text{Cov}(Z_{t-1}, Y_t) \\ &= 0.7\gamma_1 + \sigma^2 + 0.18\sigma^2 \qquad \text{by } (\textbf{??}) \text{ and } (\textbf{??}) \\ &= 0.7\gamma_1 + 1.18\sigma^2 \end{aligned} \tag{6}$$

- taking the covariance with Y_{t-1},

$$\begin{aligned}\gamma_1 = \text{Cov}(Y_t, Y_{t-1}) &= 0.7\text{Cov}(Y_{t-1}, Y_{t-1}) + \text{Cov}(Z_t, Y_t) \\ &\quad +0.2\text{Cov}(Z_{t-1}, Y_{t-1}) \\ &= 0.7\gamma_0 + 0.2\sigma^2 \qquad \text{by } (\textbf{??}) \end{aligned} \tag{7}$$

Solving (**??**) and (**??**) simultaneously, we obtain

$$\gamma_0 = \frac{44}{17}\sigma^2 \qquad \text{and} \qquad \gamma_1 = \frac{171}{85}\sigma^2$$

Hence

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{171}{220}$$

[3]

As usual, $\rho_0 = 1$. [1]

Now, using the defining equation of the model, we have that for $k \geq 2$

$$\begin{aligned}\gamma_k = \text{Cov}(Y_t, Y_{t-k}) &= 0.7\text{Cov}(Y_{t-1}, Y_{t-k}) + \text{Cov}(Z_t, Y_{t-k}) \\ &\quad +0.2\text{Cov}(Z_{t-1}, Y_{t-k}) \\ &= 0.7\gamma_{k-1}\end{aligned}$$

Hence $\rho_k = 0.7\rho_{k-1}$ and (by induction) $\rho_k = 0.7^{k-1}\rho_1$ for $k \geq 2$. [2]

c) Let $\alpha = \frac{171}{220}$ and $\beta = 0.7$. Let $z \in \mathbb{C}$ be given by $z = \cos\omega - i\sin\omega$, so that for each $k \geq 0$, $z^k + z^{-k} = 2\cos(k\omega)$. Then [1]

$$\begin{aligned}f^*(\omega) &= 1 + \sum_{k=1}^{\infty} \rho_k (z^k + z^{-k}) \\ &= 1 + \frac{\alpha z}{1-\beta z} + \frac{\alpha z^{-1}}{1-\beta z^{-1}} \\ &= \frac{(1-2\alpha\beta+\beta^2) + (\alpha-\beta)(z+z^{-1})}{(1+\beta^2) - \beta(z+z^{-1})} \\ &= \frac{(1-2\alpha\beta+\beta^2) + 2(\alpha-\beta)\cos\omega}{(1+\beta^2) - 2\beta\cos\omega}\end{aligned}$$

With the values of α and β given we obtain [3]

$$f^*(\omega) = \frac{442 + 170\cos\omega}{1639 - 1540\cos\omega}$$

[1]

d) We have that $(1-0.7B)Y_t = (1+0.2B)Z_t$. Hence, [1]

$$\begin{aligned}\pi(B) &= (1-0.7B)(1+0.2B)^{-1} \\ &= (1-0.7B)(1-0.2B+0.2^2B^2-0.2^3B^3+0.2^4B^4-\cdots) \\ &= 1-(0.2+0.7)B+(0.2^2+0.7(0.2))B^2-(0.2^3+0.7(0.2)^2)B^3 \\ &\quad +(0.2^4+0.7(0.2)^3)B^4-\cdots\end{aligned}$$

Hence, for $k \geq 1$, [2]

$$\begin{aligned}\pi_k &= -(-1)^k(0.2^k + 0.7(0.2)^{k-1}) \\ &= 0.9(-0.2)^{k-1}\end{aligned}$$

and $\pi_0 = 1$. [1]

[1]

e) The model is clearly not appropriate for the data:

- Almost all the residuals are positive, so do not show the desired randomness.
- The ACF of the residuals indicates some seasonality (autocorrelations at lags $3, 6, 9, \ldots$ are positive). This is not surprising given the nature of the data. [3]

[Total Q3: 20]

4. a) (i) $(1-\alpha B)(1-B)^2 Y_t = (1+\beta B)Z_t$ and hence, expanding the terms in brackets,

$$Y_t = (2+\alpha)Y_{t-1} - (1+2\alpha)Y_{t-2} + \alpha Y_{t-3} + Z_t + \beta Z_{t-1}$$

Replacing t by $t+k$ we have that [1]

$$Y_{t+k} = (2+\alpha)Y_{t+k-1} - (1+2\alpha)Y_{t+k-2} + \alpha Y_{t+k-3} + Z_{t+k} + \beta Z_{t+k-1}$$

Setting $k=1$ and taking conditional expectations up to time t, we obtain [1]

$$Y_t(1) = (2+\alpha)Y_t - (1+2\alpha)Y_{t-1} + \alpha Y_{t-2} + \beta Z_t$$

Now letting $k \geq 2$ and again taking conditional expectations up to time t, [1]

$$Y_t(k) = (2+\alpha)Y_t(k-1) - (1+2\alpha)Y_t(k-2) + \alpha Y_t(k-3)$$

[1]

(ii) We have the LDE $(1-\alpha B)(1-B)^2 Y_t(k) = 0$, which has roots $1, 1, \alpha$. [1]

Hence the forecast has the form $Y_t(k) = (A_t + B_t k)1^k + C_t \alpha^k$, as required. [1]

(iii)

$$\begin{aligned} Y_t &= (1-\alpha B)^{-1}(1-B)^{-2}(1+\beta B)Z_t \\ &= (1+\alpha B+\cdots)(1+2B+\cdots)(1+\beta B)Z_t \\ &= (1+(2+\alpha+\beta)B+\cdots)Z_t \end{aligned}$$

Hence, $\psi_1 = 2+\alpha+\beta$. [1]

b) (i) From the R output, $\widehat{\alpha} = -0.6934$ and $\widehat{\beta} = -0.4292$. Hence the fitted model is [1]

$$(1+0.6934B)(1-B)^2 Y_t = (1-0.4292B)Z_t$$

with $\widehat{\sigma}_Z^2 = 1.106$. [2]

(ii) Letting $t = 140$,

$$\begin{aligned} y_{140}(0) &= y_{140} = 5937.000 = A_t + C_t & (8) \\ y_{140}(1) &= 5942.240 = A_t + B_t + \widehat{\alpha}C_t & (9) \\ y_{140}(2) &= 5948.215 = A_t + 2B_t + \widehat{\alpha}^2 C_t & (10) \end{aligned}$$

From (**??**) we have that [2]

$$A_t = 5937.000 - C_t \tag{11}$$

From (**??**) and (**??**):

$$5.24 = B_t - 1.6934C_t \tag{12}$$

From (**??**) and (**??**):

$$11.215 = 2B_t - 0.5192C_t \tag{13}$$

Solving (**??**) and (**??**) simultaneously, we obtain

$$B_t = 5.6740 \qquad \text{and} \qquad C_t = 0.2563$$

From (**??**), $A_t = 5936.74$. Substituting this into the solution of part a(ii) gives the required expression. [2]

(iii) Lag 1: the standard error is $\sqrt{\sigma_Z^2} = \sqrt{1.106} = 1.052$. Hence the 95% prediction limits are [1]

$$5942.240 \pm 1.96(1.052) = (5940.18, 5944.30)$$

Lag 2: the standard error is $\sqrt{(1+\psi_1^2)\sigma_Z^2} = 1.399$ Hence the 95% prediction limits are [1] [1]

$$5948.215 \pm 1.96(1.399) = (5945.41, 5950.89)$$

[1]

(iv) The standard error increases without bound as the lag $k \to \infty$. [2]

[Total Q4: 20]